



TITLE:

# Studies on the State and Parameter Estimation for Discrete-Time Stochastic Systems( Dissertation\_全文 )

AUTHOR(S):

Katayama, Tohru

---

CITATION:

Katayama, Tohru. Studies on the State and Parameter Estimation for Discrete-Time Stochastic Systems. 京都大学, 1969, 工学博士

ISSUE DATE:

1969-05-23

URL:

<https://doi.org/10.14989/doctor.k904>

RIGHT:

STUDIES ON THE  
STATE AND PARAMETER ESTIMATION FOR  
DISCRETE - TIME STOCHASTIC SYSTEMS

Tohru Katayama

January 1969

STUDIES ON THE STATE AND PARAMETER ESTIMATION

FOR

DISCRETE-TIME STOCHASTIC SYSTEMS

Dissertation

Submitted in partial fulfilment of the

requirements for the degree of

Doctor of Philosophy

in

Applied Mathematics and Physics

by

Tohru Katayama

Kyoto University

Kyoto, Japan

January, 1969

#### ACKNOWLEDGMENTS

The author should like to express his sincere appreciation to Professor Yoshikazu Sawaragi for his guidance and encouragement during the course of the author's graduate works and thesis reserches. The author is also grateful to Professor Yoshifumi Sunahara, Kyoto Institute of Technology, who read the manuscript and made helpful suggestions and improvements. The author's many thanks should be extended to Professor Hidekatsu Tokumaru, Dr. Hiroshi Fukawa, Dr. Toshiro Ono, and members of Professor Sawaragi's laboratory for their stimulating discussions.

All the numerical calculations are carried out at the Kyoto University Computation Center, and at the University of Tokyo Computation Center.

TABLE OF CONTENTS	Page
ACKNOWLEDGMENTS . . . . .	ii
ABSTRACT. . . . .	viii
CHAPTER 1 INTRODUCTION. . . . .	1
1.1 Introduction and Historical Remarks . . . . .	1
1.2 Outline of the Dissertation . . . . .	4
1.3 Notation Conventions. . . . .	5
1.4 Stochastic Estimation Problem . . . . .	6
1.4.1 Some Definitions of Stochastic Processes. . . . .	6
1.4.2 Bayesian Approach to Estimation Problem . . . . .	7
1.4.3 Estimation Problem for Stochastic Processes . . . . .	8
1.4.4 Error Criteria. . . . .	10

## CHAPTER 2 PERFORMANCE LOSS FOR SEQUENTIAL STATE ESTIMATION

WITH ERRORS IN NOISE COVARIANCE MATRICES. . . . .	14
2.1 Introduction. . . . .	14
2.2 Fundamental Equations of the Discrete-Time Wiener-Kalman Filter. . . . .	15
2.3 Derivation of Error Matrices. . . . .	20
2.4 Numerical Examples. . . . .	23
2.4.1 Stationary Performance Loss for a First Order System. . . . .	23
2.4.2 Nonstationary Performance Loss for a Second Order System . . . . .	27
2.5 Performance Loss in Regulator Problem . . . . .	29
2.6 Concluding Remarks. . . . .	33

## CHAPTER 3 PERFORMANCE LOSS FOR SEQUENTIAL STATE ESTIMATION

WITH ERRORS IN SYSTEM PARAMETERS. . . . .	34
3.1 Introduction. . . . .	34
3.2 Statement of Problem. . . . .	35
3.3 Derivation of Actual Error Covariance Matrix. . . . .	37
3.4 Qualitative Aspects of Performance Loss for a First Order System. . . . .	41
3.5 Determination of a Suboptimal Gain for Linear Systems with Random Parameters. . . . .	43
3.6 Digital Simulation Studies. . . . .	48
3.7 Concluding Remarks. . . . .	52

## CHAPTER 4 ESTIMATION OF STATE VARIABLES FOR NONLINEAR

DYNAMICAL SYSTEMS . . . . .	53
-----------------------------	----

4.1	Introduction. . . . .	53
4.2	Statement of Problem. . . . .	54
4.3	A Posteriori Probability Density Function . . . . .	56
4.4	Approximation Technique for Nonlinear Systems . . . . .	59
4.5	Recursive Relations for Approximate Estimator . . . . .	63
4.6	Concluding Remarks. . . . .	67
	Appendix A. . . . .	68
	Appendix B. . . . .	69

## CHAPTER 5 ACCURACY CONSIDERATION FOR APPROXIMATE

	STATE ESTIMATION TECHNIQUE. . . . .	71
5.1	Introduction and Problem Statement. . . . .	71
5.2	Approximate State Estimation Scheme . . . . .	72
5.3	Exact Expression of the a Posteriori Probability Density Function. . . . .	74
5.4	Digital Simulation Studies. . . . .	75
5.4.1	Nonlinear System with Linear Observation. . . . .	75
5.4.2	Linear System with Nonlinear Observation. . . . .	79
5.5	Concluding Remarks. . . . .	81

## CHAPTER 6 ESTIMATION OF UNKNOWN PARAMETERS FOR

	NONLINEAR DYNAMICAL SYSTEMS . . . . .	83
6.1	Introduction. . . . .	83
6.2	Statement of Problem. . . . .	85
6.3	A Posteriori Probability Density Function and General Discussion. . . . .	87

6.4	Reformulation of the Problem. . . . .	93
6.5	Approximate Nonlinear Estimator . . . . .	94
6.6	Digital Simulation Studies. . . . .	97
6.6.1	Numerical Results for a First Order System. . . . .	97
6.6.2	Numerical Results for a Second Order System . . . . .	102
6.7	Concluding Remarks. . . . .	107

## CHAPTER 7 ACCURACY CONSIDERATION FOR APPROXIMATE

	PARAMETER ESTIMATION TECHNIQUE. . . . .	108
7.1	Introduction and Problem Statement. . . . .	108
7.2	Approximate Parameter Estimation Scheme . . . . .	109
7.3	Exact Expression of the a Posteriori Probability Density Function. . . . .	110
7.4	Digital Simulation Studies. . . . .	112
7.5	Concluding Remarks. . . . .	117

## CHAPTER 8 ESTIMATION METHOD OF SLOWLY TIME-VARYING PARAMETERS

	AND ITS APPLICATION TO ADAPTIVE CONTROL . . . . .	119
8.1	Introduction. . . . .	119
8.2	Statement of Problem. . . . .	120
8.3	Estimation Method for Time-Varying Parameters . . . . .	122
8.4	Application to Adaptive Control in the Presence of Noise. . . . .	126
8.5	Examples of Adaptive Control. . . . .	127
8.5.1	Second Order System with an Unknown Constant Parameter. . . . .	127



8.5.2 First Order System with a Slowly Time-Varying Parameter. . . . .	133
8.6 Concluding Remarks. . . . .	139
REFERENCES. . . . .	141

STUDIES ON THE STATE AND PARAMETER ESTIMATION  
FOR  
DISCRETE-TIME STOCHASTIC SYSTEMS

by

Tohru Katayama

Doctor of Philosophy in Applied Mathematics and Physics

Kyoto University, Kyoto, Japan, 1969

Professor Yoshikazu Sawaragi, Chairman

ABSTRACT

This dissertation is mainly divided into three parts.

The first part is concerned with the problem of performance loss in the linear sequential state estimation due to incorrect knowledge of noise covariance matrices and system parameters. The optimal, calculated, and actual covariance matrices of estimation errors are derived; from these covariance matrices, the performance losses are evaluated. The relation between the sequential state estimation and the noise free regulator problem is also investigated from the performance loss point of view.

The second part is devoted to the state estimation for nonlinear dynamical systems. An approximation technique of estimating state

variables is derived on the basis of the assumption that the a posteriori probability density function of the state conditioned on observed data is Gaussian. The accuracy of the approximate state estimation technique is also studied from the Bayesian point of view, including digital simulation studies.

The third part considers the parameter estimation problem for nonlinear dynamical systems. General discussion of the parameter estimation problem is given from the Bayesian point of view. It is clearly shown that the parameter estimation can not be separated from the state estimation when the dynamical system is subjected to random disturbance and the output observation is corrupted by noise. In view of this, the parameter estimation problem is successfully embedded in the state estimation problem for nonlinear dynamical systems. The accuracy consideration is also carried out for a simple first order system. The problem of estimating slowly time-varying parameters is also treated; a method of the modification of the scheme of estimating unknown constant parameters is introduced. The resultant scheme is applied to the adaptive compensation of control performance of the system with unknown parameters.

## CHAPTER 1

### INTRODUCTION

#### 1.1 Introduction and Historical Remarks

Modern control theory is a direct outcome of the desire to put the automatic control theory into a rigorous mathematical framework, and is based on the concept of "state" and matrix theory[52]. Hence, the modern control theory starts with the characterization of systems by state variables. To develop a practical consideration, it is necessary to assume that not all the state variables are accessible for direct measurement and that the observed state variables may be obscured by errors caused by the external disturbances and/or the inaccuracy of measurement devices. Furthermore, since in many control processes encountered in practice there are such various uncertainties as parameter variations, incomplete knowledge of process dynamics, etc., the design of optimal

control systems requires the concept of stochastic or adaptive control processes[5]. In this respect, the estimation of state variables and unknown parameters from noisy data forms the important problems in the field of stochastic and adaptive control.

The basic contributions to the problem of estimating state variables for noisy linear dynamical systems essentially date back to Kolmogorov[28], Wiener[53], and Doob[12], wherein the problem is solved for stationary random signals and noises.

At the earlier stage, Zadeh & Ragazzini[58], Booton[7], and Steeg [49], etc., developed the optimal estimation problem for nonstationary stochastic processes as the extensive studies of the Wiener's theory. Because of the difficulty of solving the so-called Wiener-Hopf integral equation, the nonstationary problem remained unsolved, until the appearance of the works by Kalman[23], and Kalman & Bucy[27].

Kalman and Bucy formulated the nonstationary problem by using the concept of "state" and differential (or difference) equation, and converted the Wiener- Hopf integral equation into a set of differential (or difference) equations. Therefore, the resulting optimal filter, which is usually referred to as the Wiener-Kalman filter, can be mechanized by an analog or digital computer; the output gives the desired solution to the optimal estimation problem.

Since these works, numerous investigation and application followed. Recently, the problem of performance loss (or mismatch) of the Wiener-Kalman filter due to errors in initial statistics was treated by Soong [46] for a single stage estimation problem. Nishimura[37] extended the

result to the multistage sequential estimation problem. There are also related works by Nishimura[38] and Neal[36], who considered the performance loss due to errors in noise covariance matrices and in system dynamics, respectively. Sorenson[47] presented a method of evaluating the performance bounds of the Wiener-Kalman filter.

Another important problem is the nonlinear state estimation problem, which was first treated by Zadeh[57]. The solution of the nonlinear estimation problem follows from the a posteriori probability distribution (or density) function of the signal process conditioned on observed data. The fundamental papers of deriving a partial differential equation of the a posteriori probability density function are due to Stratonovich[50], Kushner[31], Wonham[55] and Bucy[8]. The approximate methods of solving the nonlinear estimation problem were developed by Bass et al.[4], Kushner[32] for continuous-time systems, and by Cox[9] and Sorenson & Stubberud[48] for discrete-time systems.

On the other hand, the parameter estimation problem from noisy data has its root in the early least square differential-correction scheme by Gauss[17] in connection with orbits determination of the planets. Until the present, a number of techniques for the identification or the parameter estimation of linear dynamical systems have been reported; some of the important contributions are due to Kalman[22], and Levin[34], who treated the problem from the viewpoint of least square; Kushner[30] considered a computationaly feasible recursive scheme. A recent review of the process parameter estimation was presented by Eykhoff et al[14].

The nonlinear identification problems were first developed by

Wiener[54] and later by Balakrishnan[3], etc. Cox[9] and Detchmندی & Sridhar[11] treated the state and parameter estimation problem by using dynamic programming and invariant embedding, respectively. There are also related works by Fukao[16] and Kumar[29], who applied the Bayesian learning and the quasi-linearization technique, respectively.

## 1.2 Outline of the Dissertation

The main problems considered in this dissertation are as follows.

- (i) The performance loss of the discrete-time Wiener-Kalman filter designed on the basis of erroneous mathematical model[43, 44].
- (ii) The state estimation problem for discrete-time nonlinear stochastic systems[40].
- (iii) The parameter estimation problem for discrete-time nonlinear stochastic systems[41, 42].

In Chapter 2, the performance losses in the sequential estimation problem due to incomplete knowledge of the noise covariance matrices are evaluated. Chapter 3 deals with the problem of performance loss due to the existence of parameters, unknown except for their nominal values, in both transition and observation matrices; the result is applied to the design of linear filters for systems with random parameters. The relation between the estimation problem and the noise free regulator problem is also treated in these chapters from the viewpoint of performance loss.

Chapter 4 is devoted to the derivation of the approximate solution of the nonlinear state estimation problem. The principal approach is

based on the Bayesian estimation method and the application of the concept of statistical linearization technique; the key assumption involved is that the a posteriori probability density function of the state conditioned on the observed data is Gaussian. In Chapter 5, the accuracy of the approximate state estimation technique is examined by means of digital simulation studies.

Chapter 6 is concerned with the nonlinear parameter estimation problem. The approximation method is derived by extending the result in Chapter 4. Numerical versions are also demonstrated and showed excellent tracking behavior for unknown parameters. The accuracy consideration similar to that of Chapter 5 is carried out in Chapter 7; the 2-dimensional joint a posteriori probability density function of state and unknown parameter is evaluated by a digital computer.

Finally, Chapter 8 considers the estimation of slowly time-varying parameters. The method of the modification of the error covariance matrix is presented by extending the technique in Chapter 6 to the present situation. The resultant scheme is applied to the adaptive compensation of the control performance of the system with unknown parameters, including several numerical results.

### 1.3 Notation Conventions

Throughout this dissertation, we shall be concerned with discrete-time (or sampled data) dynamical systems; the signal may be observed at equally spaced sampling instants. The sampling period can be chosen



unity without loss of generality. The variables denoting the time, such as  $j, k, k_1$ , are always integers. The symbols  $x, u, v, \dots, z, \theta$  denote column vectors with components  $x_i, u_i, v_i, \dots, z_i, \theta_i$ . The capital letters  $F, H, \dots, \Phi, \Gamma$  denote matrices whose elements are  $F_{ij}, H_{ij}, \dots, \Phi_{ij}, \Gamma_{ij}$ . The unit matrix is  $I$ . The letters  $i, j, k, m, n, \ell, N$  are arbitrary integers. The transpose of a matrix or a vector is represented by the prime ('). The inner product of  $x$  and  $y$  are  $x'y$ , and the outer product is the matrix  $xy'$  with components  $x_i y_j$ . The norm  $\|x\|$  of a vector  $x$  is  $\sqrt{x'x}$ . If  $A$  is a symmetric and nonnegative definite matrix, we use the abbreviation  $\|x\|_A^2$  for the quadratic form  $x'Ax$ . The symbol  $E\{\cdot\}$  denotes the mathematical expectation,  $E\{\cdot|\cdot\}$  denotes the conditional expectation, and  $p(\cdot|\cdot)$  is the conditional probability density function.

Numbers within the brackets indicate the references at the end of this monograph, which are arranged in alphabetical order.

#### 1.4 Stochastic Estimation Problem

In order to make the class of problems to be studied clear, we briefly describe some basic concepts of stochastic (or random) process and the associated optimal estimation problems.

##### 1.4.1 Some Definitions of Stochastic Processes[Doob, 12]

A stochastic process is a family of functions  $\{x(t, \omega)\}$  which depends upon two arguments  $t$  and  $\omega$ ;  $t$  is the time and a real number, and  $\omega$  is a random event and is a point in an abstract space  $\Omega$ . If  $t_0$  is a fixed

time,  $x(t, \omega)$  is a random variable, which is a numerical function of the point  $\omega$  in  $\Omega$ . If the value of the argument  $\omega$  is fixed, say  $\omega_0$ , then we have a real function  $x(t, \omega_0)$  of time. This function is referred to as a sample function or a realization of the stochastic process, and letting  $\omega_0$  take all values in  $\Omega$ , we get a collection of such functions.

Let  $t_1, \dots, t_n$  be any finite set of parameter values of the process. The distribution of the process is characterized by the multi-dimensional distribution of the random variables  $x(t_1, \omega), \dots, x(t_n, \omega)$ . If this multi-dimensional distribution is Gaussian, the process is called Gaussian.

If we restrict time  $t$  to be integers, say  $k$ , the family  $\{x(k, \omega)\}$  is called a discrete-time stochastic process or stochastic sequence. In this monograph, we shall mainly be concerned with discrete-time stochastic processes. We often simply write  $x(k)$  instead of  $\{x(k, \omega)\}$ , suppressing the dependence on the argument  $\omega$ .

#### 1.4.2 Bayesian Approach to Estimation Problem[18, 33]

One of the basic stochastic estimation method is the Bayesian approach. Suppose now that the statistical parameter  $\theta$  and the observation  $z$  are somehow related, where  $\theta$  is an  $m$ -dimensional random vector and  $z$  is a  $p$ -dimensional vector. The parameter  $\theta$  can be regarded as unknown parameters or state variables of a dynamical system. In the former case, we have a parameter estimation or identification problem, and in the latter case it is a state estimation problem.

The Bayesian approach is based on the a posteriori probability density function  $p(\theta|z)$  of the parameter  $\theta$  conditioned on observation  $z$ . By

using the Bayes theorem

$$p(\theta|z) = \frac{p(z|\theta)p(\theta)}{\int p(z|\theta)p(\theta)d\theta} \quad (1.1)$$

where  $p(z|\theta)$  is the conditional probability density function of  $z$  given  $\theta$ , and  $p(\theta)$  is the a priori probability density function of  $\theta$ .

Since the a posteriori probability density function contains all the information about the parameter  $\theta$ , if we know the a posteriori probability density function  $p(\theta|z)$ , we can readily derive various estimates.

- (i) The most probable estimate; this is the mode of the a posteriori probability density function.
- (ii) The estimate that minimizes the maximum possible error; that is, the median of the a posteriori probability density function.
- (iii) The minimum variance estimate; this is given by the mean of the a posteriori probability density function.

Note that if the  $p(\theta|z)$  is Gaussian, all these estimates coincide.

There are also familiar methods of estimation such as "Least square method" and "Maximum likelihood estimation"[10]. However, in this monograph, we shall use the Bayesian approach.

The maximum likelihood estimate coincides with the most probable estimate of the Bayesian approach, if the a priori density  $p(\theta)$  is uniform, that is, we have no a priori information about  $\theta$  except for the bounds. The least square method is powerfull when we have no information about the underlying probabilistic law.

#### 1.4.3 Estimation Problem for Stochastic Processes

Now let us consider the state estimation problem for stochastic processes on the basis of the above discussion. Since the idea of state estimation and parameter estimation is quite similar, the parameter estimation problem is not considered. We shall first be concerned with scalar-valued processes, and later a few comments will be given for vector-valued processes.

Let two different stochastic processes  $\{x(k)\}$  and  $\{v(k)\}$  are given; the  $x(k)$  is a signal process to be estimated and may be an output of a dynamical system of the form

$$x(k+1) = f(x(k), k) + w(k), \quad (1.2)$$

where  $w(k)$  is a white noise process, and  $f$  is a some function of  $x(k)$  and  $k$ . Moreover, suppose that the observation is made on the process  $z(k)$  in the form

$$z(k) = x(k) + v(k), \quad (1.3)$$

or more generally,

$$z(k) = h(x(k), k) + v(k), \quad (1.4)$$

where  $v(k)$  is the observation noise, and  $h$  is a some function of  $x(k)$  and  $k$ .

When we have observed values of the process  $\{z(k)\}$ , say  $z(0), \dots, z(k)$ , what can we infer from these data about the signal  $x(k_1)$  at time instant  $k_1$ ? If  $k_1 > k$ , this is a prediction, if  $k_1 = k$ , this is called a filtering problem, and if  $k_1 < k$ , we have a data smoothing problem. Kalman[23, 26] used the term "estimation" for these problems. In this monograph, we shall be concerned with the cases where  $k_1 = k+1$  and

$k_1 = k$ , and if there is no confusion, we shall use the terms "filtering" and "estimation" as the same meaning.

For any observed values  $z(0), \dots, z(k)$ , a realization of  $\{z(k)\}$ , we can in principle determine the probability of simultaneous occurrence of value  $\xi(k_1)$  of the random variable  $x(k_1)$ . This is characterized by the conditional distribution function

$$F(\xi) = P\{x(k_1) \leq \xi \mid z(0), \dots, z(k)\}. \quad (1.5)$$

If the probability density functions of the processes  $w(k)$  and  $v(k)$  are given, the a posteriori probability density function can be computed from Eq. (1.1).

If both the  $\{x(k)\}$  and  $\{z(k)\}$  are Gauss-Markov processes, the determination of the conditional probability distribution function is quite simple, because the conditional distribution becomes Gaussian and is completely characterized by its mean and covariances. Furthermore, if the process is also Markovian, it suffices to know its mean and covariances at one instant of time.

As was suggested by Stratonovich[50, 51], any estimate of  $x(k_1)$  will be some function of this conditional distribution function, so that it becomes a function of random variables  $z(0), \dots, z(k)$ .

#### 1.4.4 Error Criteria

Now we introduce the loss function to select the optimal estimate. Let  $\hat{x}(k_1|k)$  be an estimate of  $x(k_1)$ ; the mathematical definition of  $\hat{x}(k_1|k)$  will be given below. The loss function is a scalar function of estimation error  $\epsilon$ , where  $\epsilon = x(k_1) - \hat{x}(k_1|k)$ , and has the form

$$\left. \begin{aligned} L(0) &= 0 \\ L(\epsilon_2) &\geq L(\epsilon_1) \geq 0, \text{ where } \epsilon_2 \geq \epsilon_1 \geq 0 \\ L(-\epsilon) &= L(\epsilon) \end{aligned} \right\} \quad (1.6)$$

The functions  $a\epsilon^2$ ,  $a\epsilon^4$ ,  $a|\epsilon|$ , etc., are examples of loss functions, where  $a \geq 0$ .

The best estimate  $\hat{x}(k_1|k)$  of the random variable  $x(k_1)$  is determined so as to minimize the expected loss

$$E\{L[x(k_1) - \hat{x}(k_1|k)]\} = E\{E\{L[x(k_1) - \hat{x}(k_1|k)]|Z_k\}\}, \quad (1.7)$$

where  $Z_k = \{z(0), \dots, z(k)\}$ . It is clear from the definition of the iterated conditional expectation[12] that the minimization of Eq. (1.7) is equivalent to the minimization of the conditional expectation

$$E\{L[x(k_1) - \hat{x}(k_1|k)]|Z_k\}, \quad (1.8)$$

because, as mentioned above, any estimate of  $x(k_1)$  is a function of the random variables  $z(0), \dots, z(k)$ .

We have theorems concerning the optimal estimation problem.

Theorem 1[Sherman, 45]

If the loss function is of the form given by Eq. (1.6), and if the conditional distribution function defined by Eq. (1.5) is

(i) symmetric about the mean  $\bar{\xi}$ ,

$$F(\xi - \bar{\xi}) = 1 - F(\bar{\xi} - \xi)$$

(ii) convex for  $\xi \leq \bar{\xi}$ ,

$$F(\lambda\xi_1 + (1-\lambda)\xi_2) \leq \lambda F(\xi_1) + (1-\lambda)F(\xi_2)$$

for all  $\xi_1, \xi_2 \leq \bar{\xi}$ , and  $0 \leq \lambda \leq 1$ ,

then, the optimal estimate which minimizes the expected loss is given by the conditional expectation

$$\hat{x}(k_1|k) = E\{x(k_1)|Z_k\}. \quad (1.9)$$

This estimate is the type (iii) of the Bayesian estimate. In particular, if the stochastic processes  $\{x(k)\}$ ,  $\{w(k)\}$  and  $\{z(k)\}$  are Gaussian, the above conditions (i) and (ii) are satisfied, so that the optimal estimate is given by the conditional expectation defined by Eq. (1.9).

Theorem 2[Doob, 12]

If the loss function is

$$L(\epsilon) = \epsilon^2, \quad (1.10)$$

then Theorem 1 holds without any assumption for the conditional distribution function of Eq. (1.5)

When the processes are vector-valued, the estimation problem can be stated in a similar way as follows.

Given vector-valued observed random variables  $z(0), \dots, z(k)$ , find the estimate  $\hat{x}(k_1|k)$  of a vector-valued random variable  $x(k_1)$ , which minimizes the expected loss

$$E\{L[\|x(k_1) - \hat{x}(k_1|k)\|^2]\}, \quad (1.11)$$

where  $\|\cdot\|$  is the norm.

Theorem 1 remains valid in the vector case, if the conditional probability distribution function of the  $n$ -dimensional vector random variable  $x(k_1)$ ,

$$F(\xi_1, \dots, \xi_n) =$$

$$= P\{x_1(k_1) \leq \xi_1, \dots, x_n(k_1) \leq \xi_n | z(0), \dots, z(k)\} \quad (1.12)$$

is symmetric with respect to the  $n$  variables  $\xi_1 - \bar{\xi}_1, \dots, \xi_n - \bar{\xi}_n$ , and is convex in the region where all these variables are negative.

In this monograph, we shall be concerned with the case where the loss function is of the quadratic form

$$E\{\|x(k_1) - \hat{x}(k_1|k)\|^2\}. \quad (1.13)$$

The problem is then called the minimum variance estimation, and the solution is given by Eq. (1.9) from Theorem 2; the main task is the derivation of the a posteriori probability density function by using the Bayes theorem.

It should be noted that the conditional expectation given by Eq. (1.9) is also the optimal solution for the loss function of the form

$$L(\epsilon) = \|\epsilon\|_P^2 \quad (1.14)$$

where  $P$  is an arbitrary  $n \times n$  symmetric nonnegative definite matrix (see Kalman[26]).

Therefore, the conditional expectation becomes the optimal solution for many loss functions.



## CHAPTER 2

### PERFORMANCE LOSS FOR SEQUENTIAL STATE ESTIMATION WITH ERRORS IN NOISE COVARIANCE MATRICES

#### 2.1 Introduction

The problem of estimating the state variables of a linear dynamical system from observed data, which is a linear combination of the state variables corrupted by additive noise, has much attention during the past several years. It is well-known that the Wiener-Kalman filter[23, 27], or the Bayesian estimation method[18] gives the unbiased minimum variance estimate for this problem. The theory has received extensive investigation and application, which are adaptive space navigation, estimation of state of control systems, plant identification for adaptive control, orbit determination, etc.

In such cases, the optimality of the sequential estimation process

depends upon the knowledge of system characteristics; for example, the a priori statistics of initial conditions, the statistics and the values of system parameters. In practice, these system characteristics are rarely known exactly, since it is usually impossible to theoretically determine these characteristics for complex physical situations. That is, the model of the physical system is usually associated with errors or lack of sufficient accuracy. The erroneous model may cause the degradation of the estimator performance. Hence, it is an important problem to analyze in advance the effect of errors in system characteristics on the sequential estimation process.

In relation to the problem mentioned above, Soong[46] and Nishimura [37] have treated the effect of errors in the initial covariance matrix of the estimation error on the filter performance.

In this chapter, we shall consider the performance loss due to incorrect information of both the system and observation noises. The remainder is devoted to the investigation of the mutual relations between the sequential estimation and the noise free regulator problem[25] from the viewpoint of performance loss.

The next chapter will be concerned with the problem of performance loss due to errors in system parameters.

## 2.2 Fundamental Equations of the Discrete-Time

### Wiener-Kalman Filter

In this section, we briefly discuss the discrete-time Wiener-Kalman filter[26], which will frequently be used in the following chapters.

Consider a system described by the linear difference equation

$$x(k+1) = \Phi(k)x(k) + w(k), \quad (2.1)$$

where

$x(k)$  is an  $n$ -dimensional state vector,

$w(k)$  is an  $n$ -dimensional vector denoting the random disturbance,

$\Phi(k)$  is an  $n \times n$  state transition matrix.

Let the output observation be linear combination of the state variables corrupted by additive noise, i.e.,

$$z(k) = H(k)x(k) + v(k), \quad (2.2)$$

where

$z(k)$  is a  $p$ -dimensional vector denoting the observed signal ( $p \leq n$ ),

$v(k)$  is a  $p$ -dimensional vector denoting the observation noise,

$H(k)$  is a  $p \times n$  observation matrix.

It is assumed that the initial state  $x(0)$  is Gaussian and is independent of the  $w(k)$  and  $v(k)$ , and that the processes  $w(k)$  and  $v(k)$  are mutually independent Gaussian white noises with means zero and covariance matrices

$$\left. \begin{aligned} E\{w(k)w'(j)\} &= Q(k)\delta_{jk} \\ E\{v(k)v'(j)\} &= R(k)\delta_{jk} \\ E\{w(k)v'(j)\} &= 0 \end{aligned} \right\} \text{ for } j, k = 0, 1, \dots, \quad (2.3)$$

where  $E$  is the operator of the mathematical expectation,  $\delta_{jk}$  is the Kronecker's delta function, the prime(') denotes the transpose of a vector or a matrix, and  $Q(k)$  and  $R(k)$  are  $n \times n$  and  $p \times p$  symmetric, nonnegative definite matrices, respectively. Fig. 2.1 shows the schematic representation of the system to be considered.

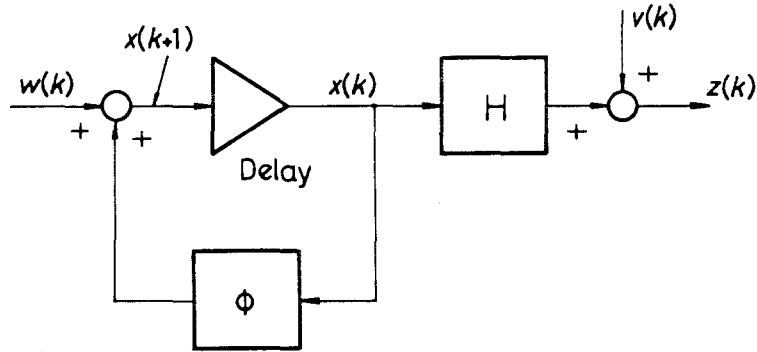


Fig. 2.1 Schematic representation of the system to be considered

The estimation problem considered here is to find the optimal estimate of the state vector  $x(k_1)$  on the basis of the observed data  $Z_k = \{z(0), \dots, z(k)\}$ . The criterion function used is the expectation of the quadratic loss function

$$E\{E\{\|x(k_1) - \hat{x}(k_1|k)\|^2 | Z_k\}\}, \quad (2.4)$$

so that the optimal estimate is given by the conditional expectation of  $x(k_1)$  relative to  $Z_k$ , i.e.,

$$\hat{x}(k_1|k) = E\{x(k_1) | Z_k\}. \quad (2.5)$$

As noted in Chapter 1, if  $k_1 > k$ , the estimation problem is called the optimal prediction, if  $k_1 = k$ , it is a filtering problem, and if  $k_1 < k$ , we have a data smoothing problem. In this and next chapters, we shall treat the case where  $k_1 = k+1$ , that is, the one step prediction problem.

The optimal filtering equation is first derived by Kalman[23], who used the "principle of orthogonal projection"[12], and later by Ho & Lee[18], who employed the Bayesian approach.

Defining the  $n \times n$  covariance matrix of estimation error by

$$P(k+1) = E\{[x(k+1) - \hat{x}(k+1|k)][x(k+1) - \hat{x}(k+1|k)]'\}, \quad (2.6)$$

the optimal solution is given by the following recursive equations.

(i) The running values of optimal estimate are successively produced by

$$\hat{x}(k+1|k) = \Phi(k)\hat{x}(k|k-1) + \Gamma(k)[z(k) - H(k)\hat{x}(k|k-1)]. \quad (2.7)$$

(ii) The optimal gain  $\Gamma(k)$  is given by

$$\Gamma(k) = \Phi(k)P(k)H'(k)[H(k)P(k)H'(k) + R(k)]^{-1}. \quad (2.8)$$

(iii) The covariance equation is

$$\begin{aligned} P(k+1) = & [\Phi(k) - \Gamma(k)H(k)]P(k)[\Phi(k) - \Gamma(k)H(k)]' \\ & + \Gamma(k)R(k)\Gamma'(k) + Q(k). \end{aligned} \quad (2.9)$$

The initial conditions to Eqs. (2.7) and (2.9) are given by  $\hat{x}(0|-1) = E\{x(0)\} = 0$  and  $P(0) = E\{x(0)x'(0)\}$ , respectively. The discrete-time Wiener-Kalman filter is shown in Fig. 2.2.

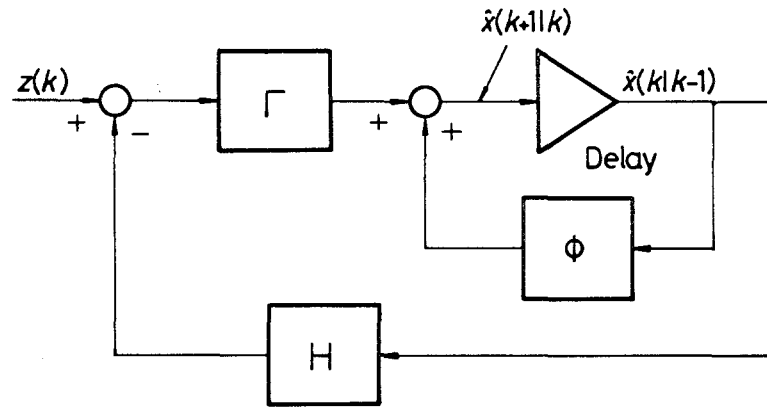


Fig. 2.2 Schematic representation of the discrete-time Wiener-Kalman filter

It should be noted that the optimal gain matrix  $\Gamma(k)$  is obtained by minimizing  $P(k+1)$  with respect to  $\Gamma(k)$ . In fact, let the right hand side of Eq. (2.9) be  $P(\Gamma)$ . Then the necessary and sufficient condition that  $P(k+1)$  is optimal is

$$P(\Gamma + \delta\Gamma) - P(\Gamma) \geq 0 \text{ (nonnegative definite)}, \quad (2.10)$$

for any perturbation  $\delta\Gamma$ ; this is equivalent to

$$\begin{aligned} \delta\Gamma[[H\Phi H' + R]\Gamma' - H\Phi\Phi'] + [\Gamma[H\Phi H' + R] - \Phi\Phi H']\delta\Gamma' \\ + \delta\Gamma[H\Phi H' + R]\delta\Gamma' \geq 0. \end{aligned} \quad (2.11)$$

Since the last term in Eq. (2.11) is nonnegative definite, the optimal gain matrix  $\Gamma(k)$  is given by Eq. (2.8), as was to be proved.

As can be seen from Eq. (2.9),  $P(k)$  does not depend upon the observed data, that is, the actual realization of the random processes. Hence,  $P(k)$  can be examined in a general manner to establish the expected behavior of the estimator before it is actually realized.

As mentioned in section 2.1, Nishimura[37] considered the effect of errors in the initial value  $P(0)$  on the sequential estimation process. Here, we extend the procedure to cover the effect of errors in noise covariance matrices  $Q(k)$  and  $R(k)$  on the filter performance. If we misidentify  $Q(k)$  and  $R(k)$  as  $Q_c(k)$  and  $R_c(k)$ , respectively, then the filter equation, resulting from Eqs. (2.7), (2.8) and (2.9) by using the incorrect covariance matrices, deviates from the optimal one. The problem considered is the analytical evaluation of the performance loss due to incorrect covariance matrices.

The same problem is also recently treated by Nishimura[38] for

continuous-time Wiener-Kalman filter. However, the discrete-time Wiener-Kalman filter is rather different from the continuous counterpart. In the continuous case[27], the covariance matrix  $R(k)$  must be positive definite to ensure the validity of the matrix Riccati equation; but in the present case, this assumption is not necessary because the filter equations do not contain the inverse  $R^{-1}(k)$ .

### 2.3 Derivation of Error Matrices

Let us introduce three kinds of covariance matrices  $P_o(k)$ ,  $P_c(k)$  and  $P_a(k)$  as follows.

$P_o$  : The optimal covariance matrix when the correct noise covariance matrices are used; this matrix is defined by Eq. (2.9).

$P_c$  : The calculated covariance matrix derived from Eq. (2.9) when the incorrect  $Q_c$  and  $R_c$  are used rather than the correct  $Q$  and  $R$ , respectively.

$P_a$  : The actual covariance matrix evaluated by the definition, when the estimate is computed by using the calculated covariance matrix.

Therefore, three error matrices are derived as their mutual differences;

$$E_{ca}(k) = P_c(k) - P_a(k), \quad (2.12)_1$$

$$E_{co}(k) = P_c(k) - P_o(k), \quad (2.12)_2$$

$$E_{ao}(k) = P_a(k) - P_o(k). \quad (2.12)_3$$

From Eq. (2.9), the calculated covariance matrix  $P_c(k)$  becomes

$$P_c(k+1) = [\Phi(k) - \Gamma_c(k) H(k)] P_c(k) [\Phi(k) - \Gamma_c(k) H(k)]' +$$

$$+ \Gamma_c(k) R_c(k) \Gamma_c'(k) + Q_c(k), \quad (2.13)$$

where

$$\Gamma_c(k) = \Phi(k) P_c(k) H'(k) [H(k) P_c(k) H'(k) + R_c(k)]^{-1}, \quad (2.14)$$

with initial condition  $P_c(0)$ .

Let  $x^*(k+1|k)$  be the output of the estimator based on the incorrect gain matrix  $\Gamma_c(k)$ ; this estimate is no longer optimal but may become suboptimal one. The suboptimal estimates are successively given by

$$x^*(k+1|k) = \Phi(k) x^*(k|k-1) + \Gamma_c(k) [z(k) - H(k) x^*(k|k-1)], \quad (2.15)$$

so that the actual estimation error,

$$e(k) = x(k) - x^*(k|k-1), \quad (2.16)$$

becomes, from Eqs. (2.1) and (2.15),

$$e(k+1) = [\Phi(k) - \Gamma_c(k) H(k)] e(k) - \Gamma_c(k) v(k) + w(k). \quad (2.17)$$

Since the actual error covariance matrix  $P_a(k)$  is defined by

$$P_a(k) = E\{e(k)e'(k)\}, \quad (2.18)$$

and since the  $w(k)$  and  $v(k)$  are mutually independent Gaussian white noises, it follows from Eq. (2.17) that

$$\begin{aligned} P_a(k+1) &= [\Phi(k) - \Gamma_c(k) H(k)] P_a(k) [\Phi(k) - \Gamma_c(k) H(k)]' \\ &\quad + \Gamma_c(k) R_c(k) \Gamma_c'(k) + Q_c(k). \end{aligned} \quad (2.19)$$

Now we proceed to the derivation of the three error matrices defined by Eq. (2.12). First, by using Eqs. (2.13) and (2.19), the difference equation for  $E_{ca}(k)$  becomes

$$E_{ca}(k+1) = [\Phi(k) - \Gamma_c(k) H(k)] E_{ca}(k) [\Phi(k) - \Gamma_c(k) H(k)]' +$$



$$+ \Gamma_c(k)[R_c(k) - R(k)]\Gamma_c'(k) + Q_c(k) - Q(k). \quad (2.20)$$

The following assertions are derived from this equation.

(A1) If  $Q_c(k) - Q(k) \geq 0$ ,  $R_c(k) - R(k) \geq 0$ , and  $E_{ca}(0) \geq 0$ , then  $E_{ca}(k) \geq 0$  for all  $k$ ; that is, the  $E_{ca}(k)$  is nonnegative definite for all  $k$ . Therefore, an upper bound for the variance of the sub-optimal estimate  $x^*(k|k-1)$  can be set, and is equal to the diagonal components of  $P_c(k)$ [38].

(A2) Conversely, if  $Q_c(k) - Q(k) \leq 0$ ,  $R_c(k) - R(k) \leq 0$ , and  $E_{ca}(0) \leq 0$ , then the  $E_{ca}(k)$  is nonpositive definite for all  $k$ .

The  $E_{ao}(k)$  is obtained by using Eqs. (2.9) and (2.13), and the matrix inversion lemma (see Chapter 4). But, since the derivation is rather lengthy, only the final result is shown[43]:

$$E_{ao}(k+1) = [\Phi(k) - \Gamma_c(k)H(k)]E_{ao}(k)[\Phi(k) - \Gamma_c(k)H(k)]' + T(k)S_c^{-1}(k)S_o(k)S_c^{-1}(k)T'(k), \quad (2.21)$$

where

$$S_o(k) = H(k)P_o(k)H(k) + R(k), \quad (2.22)$$

$$S_c(k) = H(k)P_c(k)H(k) + R_c(k), \quad (2.23)$$

and

$$T(k) = [\Phi(k) - \Gamma(k)H(k)]E_{co}(k)H'(k) + \Gamma(k)[R(k) - R_c(k)]. \quad (2.24)$$

(A3) The second term of the right hand side of Eq. (2.21) is nonnegative definite for all  $k$ , so that if  $E_{ao}(0) \geq 0$ , then  $E_{ao}(k)$  is nonnegative definite for all  $k$ . This fact is naturally expected from Eq. (2.10). Finally, the difference equation for  $E_{co}(k)$  is obtained by adding

Eq. (2.20) to Eq. (2.21):

$$\begin{aligned} E_{co}(k+1) &= [\Phi(k) - \Gamma_c(k)H(k)]E_{co}(k)[\Phi(k) - \Gamma_c(k)H(k)]' \\ &\quad + \Gamma_c(k)[R_c(k) - R(k)]\Gamma_c'(k) + Q_c(k) - Q(k) \\ &\quad + T(k)S_c^{-1}(k)S_o(k)S_c^{-1}(k)T'(k). \end{aligned} \quad (2.25)$$

(A4) Assertion similar to (A1) is derived from this equation; if

$Q_c(k) - Q(k) \geq 0$ ,  $R_c(k) - R(k) \geq 0$ , and  $E_{co}(0) \geq 0$ , then the  $E_{co}(k)$  is nonnegative definite for all  $k$ .

The further general discussion is rather difficult, so that we shall give two illustrative examples in the next section.

## 2.4 Numerical Examples

### 2.4.1 Stationary Performance Loss for a First Order System

Consider a first order system described by

$$x(k+1) = Ax(k) + w(k) \quad (2.26)$$

and

$$z(k) = x(k) + v(k), \quad (2.27)$$

where it is assumed that

$$Q(k) = q, \quad R(k) = r, \quad Q_c(k) = q_c, \quad R_c(k) = r_c.$$

Application of the general result in section 2.3 gives

$$P_o(k+1) = A^2 r P_o(k) / [r + P_o(k)] + q, \quad (2.28)$$

$$P_c(k+1) = A^2 r_c P_c(k) / [r_c + P_c(k)] + q_c, \quad (2.29)$$

$$P_a(k+1) = A^2 r_c^2 P_a(k) / [r_c + P_c(k)]^2 + A^2 r P_c^2(k) / [r_c + P_c(k)] + q, \quad (2.30)$$

$$E_{ca}(k+1) = A^2 r_c^2 E_{ca}(k) / [r_c + P_c(k)]^2 + A^2 P_c^2(k) (r_c - r) / [r_c + P_c(k)]^2 + q_c - q, \quad (2.31)$$

$$E_{ao}(k+1) = A^2 r_c^2 E_{ao}(k) / [r_c + P_c(k)]^2 + [r P_c(k) - r_c P_o(k)]^2 / \{ [r + P_o(k)] [r_c + P_c(k)]^2 \}, \quad (2.32)$$

$$E_{co}(k+1) = A^2 r r_c E_{co}(k) / \{ [r + P_o(k)] [r_c + P_c(k)] \} + (r_c - r) P_o(k) P_c(k) / \{ [r + P_o(k)] [r_c + P_c(k)] \} + q_c - q. \quad (2.33)$$

Since the parameters  $A$ ,  $q$ ,  $r$ ,  $q_c$ ,  $r_c$  are all constant, the stationary variances of the estimation errors exist†. Moreover, the rate of the convergence is quite rapid, so that, here we shall be concerned with the stationary values of estimation errors. The stationary value of  $P_o(k)$  is simply denoted by  $P_o$ , and so forth.

Fig. 2.3 displays the effect of errors in system noise variance  $q$  on the three error covariances, where the numerical values used are  $A = 0.9$ ,  $q = 1.0$ ,  $r = r_c = 1.0$ . Fig. 2.4 displays the effect of errors in observation noise variance  $r$  on the three error covariances, where  $A = 0.9$ ,  $q = 1.0 = q_c$ ,  $r = 1.0$ . In both cases, the variation of the  $E_{ao}$  is very small around its optimal value; this fact shows that, in this example, the Wiener-Kalman filter (or the Bayesian estimation method) is rather insensitive to the variations of variances of system and observation noises.

---

† The condition that assures the existence of stationary value of Riccati equation is examined by Kalman[24, 27] for continuous time systems.

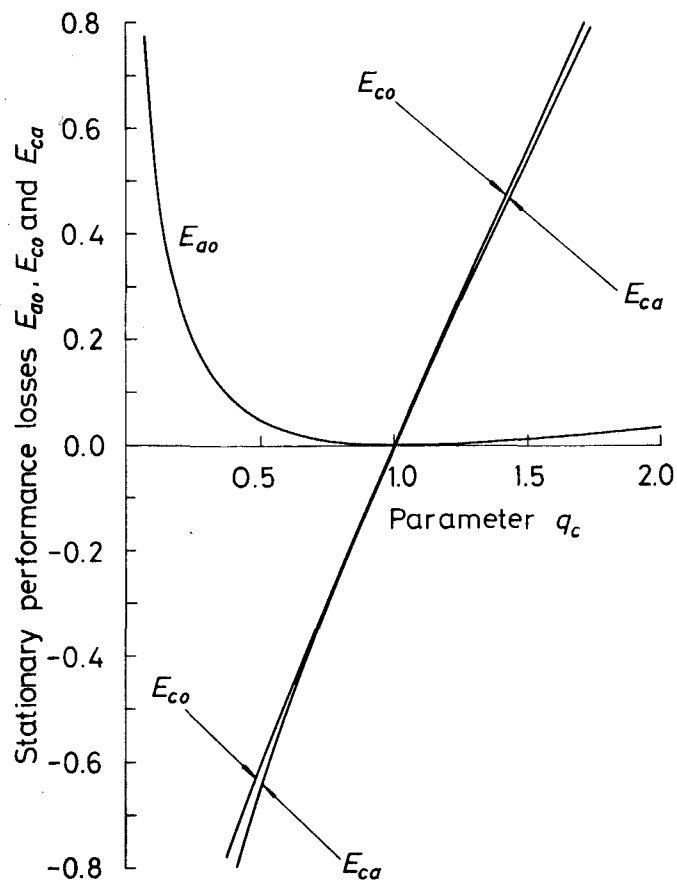


Fig. 2.3 Effect of errors in system noise variance  $q$  on the error covariances

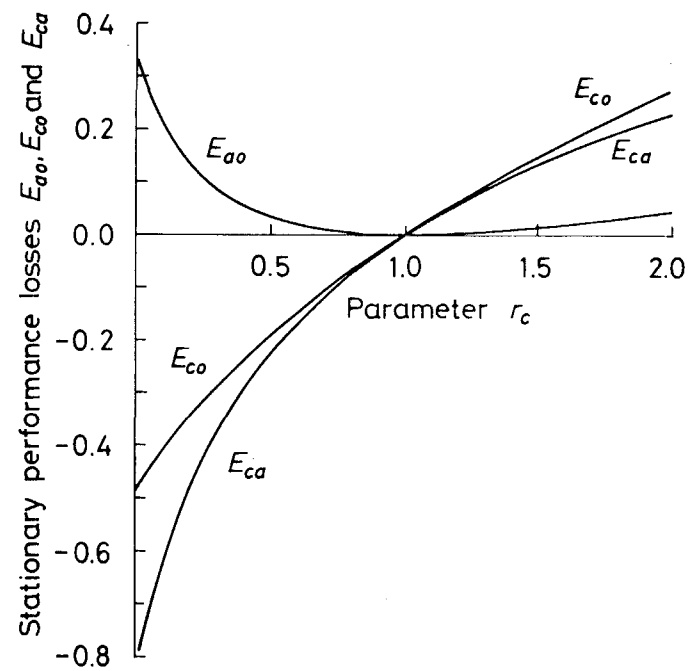


Fig. 2.4 Effect of errors in observation noise variance  $r$  on the error covariances

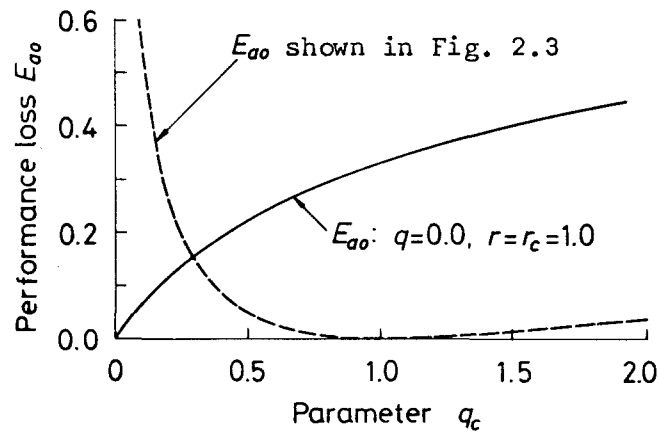


Fig. 2.5 Effect of errors in noise variance  $q$  on the error covariance  $E_{ao}$ , where  $q = 0.0$  and  $r = 1.0$

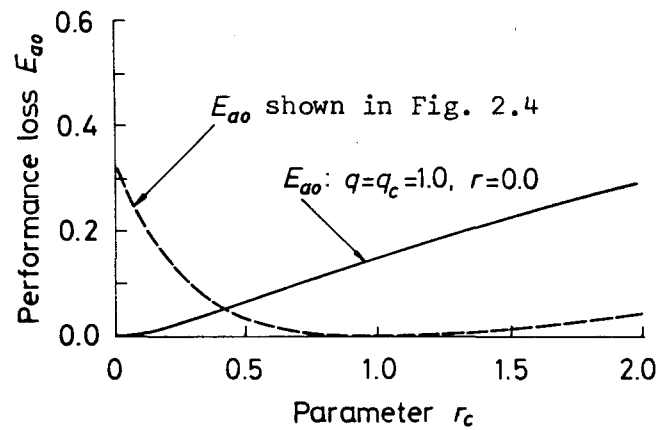


Fig. 2.6 Effect of errors in noise variance  $r$  on the error covariance  $E_{ao}$ , where  $q = 1.0$  and  $r = 0.0$

Figs. 2.5 and 2.6 display an example that the effect of errors in the system noise is quite different from that in the observation noise.

### 2.4.2 Nonstationary Performance Loss for a Second Order System

As the second example, let us consider a second order system expressed by

$$x(k+1) = \Phi(k)x(k) + w(k) \quad (2.34)$$

and

$$z(k) = H(k)x(k) + v(k), \quad (2.35)$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \quad \Phi(k) = \begin{bmatrix} 1 & T \\ A & B \end{bmatrix} \quad H(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$Q(k) = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix} \quad R(k) = r$$

Figs. 2.7 and 2.8 display the time evolution of estimation errors of state variables  $x_1(k)$  and  $x_2(k)$ , respectively. The following numerical values are used in the computation:

$$A = -0.1, \quad B = 0.95, \quad T = 0.1, \quad q = 1.0, \quad r = 1.0,$$

$$q_c = 0.6, \quad r_c = 1.4$$

$$P_o(0) = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} = P_a(0), \quad P_c(0) = \begin{bmatrix} 0.5 & 0 \\ 0 & 5 \end{bmatrix}$$

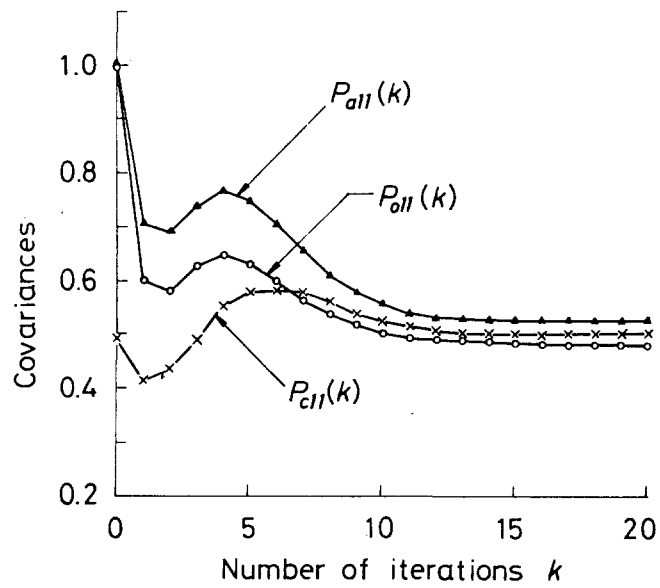


Fig. 2.7 Variation of optimal, calculated and actual covariances of the state  $x_1(k)$

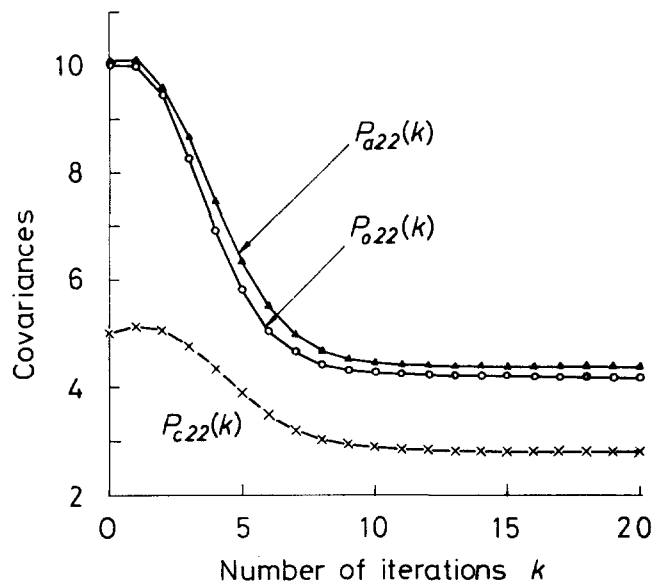


Fig. 2.8 Variation of optimal, calculated and actual covariances of the state  $x_2(k)$

## 2.5 Performance Loss in Regulator Problem

It is well-known that the optimal estimation problem is the dual of the design of optimal regulator[25]. In this section, we shall examine the mutual relations between the estimation problem and the noise free regulator problem from the viewpoint of performance loss.

Now consider a system described by the difference equation

$$x(k+1) = F(k)x(k) + G(k)u(k) \quad (2.36)$$

with a criterion function

$$I_N = \sum_{j=0}^{N-1} \{ \|x(j+1)\|_{\Lambda(j+1)}^2 + \|u(j)\|_{\Theta(j)}^2 \}, \quad (2.37)$$

where

$x(k)$  is an  $n$ -dimensional state vector, which is directly observable,

$u(k)$  is a  $p$ -dimensional control vector ( $p \leq n$ ),

$F(k)$  is an  $n \times n$  state transition matrix,

$G(k)$  is an  $p \times p$  driving matrix,

$\Lambda(k)$  is an  $n \times n$  symmetric and nonnegative definite matrix,

$\Theta(k)$  is a  $p \times p$  symmetric and nonnegative definite matrix†.

The optimal regulator problem is to find the control sequence  $\{u(j)\}$ ,

$j = 0, 1, \dots, N-1$ , which minimizes the criterion function given by

Eq. (2.37).

Assuming that there exists the optimal control sequence, we define

---

† In continuous systems, in order to find the unique solution, the  $p \times p$  matrix  $\Theta$  must be invertible; otherwise there must be some restrictions on the control  $u(k)$ , such as  $u(k)$  is in some compact subset of  $p$ -dimensional Euclidian space. In the present case, however, the existence of the inverse of  $\Theta$  is not always necessary.



$$f_{N-k}[x(k)] = \min_{\{u\}} \sum_{j=k}^{N-1} \{ \|x(j+1)\|_{\Lambda(j+1)}^2 + \|u(j)\|_{\Theta(j)}^2 \}. \quad (2.38)$$

Then applying the method of dynamic programming, we have the functional equation of the form

$$f_{N-k}[x(k)] = \min_{u(k)} \{ \|x(k+1)\|_{\Lambda(k+1)}^2 + \|u(k)\|_{\Theta(k)}^2 + f_{N-k+1}[x(k+1)] \}. \quad (2.39)$$

Assuming that

$$f_{N-k}[x(k)] = \|x(k)\|_{\Pi(k)}^2, \quad (2.40)$$

and substituting Eq. (2.40) into Eq. (2.39), we have the optimal control law

$$u_o(k) = -B(N-k)x(k), \quad (2.41)$$

where the  $p \times n$  feedback matrix  $B(N-k)$  and the recurrence relation for  $\Pi(N-k)$  is as follows[52]:

$$B(N-k) = [L_{GG}^{(N-k+1)} + \Theta(k)]^{-1} L_{GF}^{(N-k+1)} \quad (2.42)$$

and

$$\Pi(N-k) = L_{FF}^{(N-k+1)} - L_{FG}^{(N-k+1)} B(N-k) \quad (2.43)$$

with  $\Pi(0) = 0$  ( $n \times n$  matrix). The following notations are used[52].

$$\left. \begin{aligned} S(N-k+1) &= \Lambda(k+1) + \Pi(N-k+1) \\ L_{FG}^{(N-k+1)} &= F'(k)S(N-k+1)G(k) \end{aligned} \right\} \quad (2.44)$$

Rewriting Eq. (2.43), it follows that

$$S(N-k) = [F(k) - G(k)B(N-k)]' S(N-k+1) [F(k) - G(k)B(N-k)] +$$

$$+ B'(N-k)\Theta(k)B(N-k) + \Lambda(k), \quad (2.45)$$

where  $0 \leq k \leq N-1$ ; this equation is of Riccati type and is the backward equation with respect to  $k$ .

Comparing Eqs. (2.9) and (2.45), we may observe the following correspondences:

$$\left. \begin{aligned} F(k) &\leftrightarrow \Phi'(k), & G(k) &\leftrightarrow H'(k) \\ \Lambda(k) &\leftrightarrow Q(k), & B(N-k) &\leftrightarrow \Gamma(k) \\ \Theta(k) &\leftrightarrow R(k), & S(N-k) &\leftrightarrow P(k+1) \end{aligned} \right\} \quad (2.46)$$

This fact is usually referred to as the "Duality Principle" between the regulator and the filtering problem[25].

Let us now examine the performance loss due to incorrect weighting matrices  $\Lambda_c(k)$  and  $\Theta_c(k)$  rather than correct  $\Lambda(k)$  and  $\Theta(k)$ . In this case, from Eq. (2.42), the feedback matrix becomes

$$B_c(N-k) = [L_{GG}^c(N-\overline{k+1}) + \Theta_c(k)]^{-1} L_{GF}^c(N-\overline{k+1}), \quad (2.47)$$

where super- and sub-scripts(c) implies that the incorrect weighting matrices are used rather than the correct ones. Therefore, Eq. (2.45) becomes

$$\begin{aligned} S_c(N-k) &= [F(k) - G(k)B_c(N-k)]' S_c(N-\overline{k+1}) [F(k) - G(k)B_c(N-k)] \\ &\quad + B_c'(N-k)\Theta_c(k)B_c(N-k) + \Lambda_c(k), \end{aligned} \quad (2.48)$$

which really corresponds to Eq. (2.13).

When the feedback matrix  $B_c(N-k)$  is used instead of  $B(N-k)$ , the control law becomes

$$u_c(k) = -B_c(N-k)x(k). \quad (2.49)$$

Therefore, substitution of Eq. (2.49) into Eq. (2.36) gives

$$x(k+1) = [F(k) - G(k)B_c(N-k)]x(k), \quad (2.50)$$

so that the actual control performance can be evaluated as follows.

Defining

$$\gamma_{N-k}^2[x(k)] = \sum_{j=k}^{N-1} \{ \|x(j+1)\|_{\Lambda(j+1)}^2 + \|u_c(j)\|_{\Theta(j)}^2 \}, \quad (2.51)$$

we have the functional equation

$$\gamma_{N-k}^2[x(k)] = \|x(k+1)\|_{\Lambda(k+1)}^2 + \|u_c(k)\|_{\Theta(k)}^2 + \gamma_{N-k+1}^2[x(k+1)]. \quad (2.52)$$

Again assuming that

$$\gamma_{N-k}^2[x(k)] = \|x(k)\|_{\Pi_a(k)}^2, \quad (2.53)$$

and substituting Eq. (2.53) into Eq. (2.52), we obtain

$$\begin{aligned} S_a(N-k) &= [F(k) - G(k)B_c(N-k)]' S_a(N-k+1) [F(k) - G(k)B_c(N-k)] \\ &\quad + B_c'(N-k)\Theta(k)B_c(N-k) + \Lambda(k), \end{aligned} \quad (2.54)$$

where

$$S_a(N-k) = \Pi_a(N-k) + \Lambda(k). \quad (2.55)$$

Eq. (2.54) actually corresponds to Eq. (2.19); this fact shows that there still exists the duality principle in the sense of performance loss between the estimation problem and the noise free regulator problem.

Since the optimal performance is  $\|x(0)\|_{\Pi(N)}^2$  and the actual performance due to incorrect information is  $\|x(0)\|_{\Pi_a(N)}^2$ , the mutual difference can be evaluated by using the relation

$$\Pi_a(N) - \Pi(N) = S_a(N) - S(N). \quad (2.56)$$

Therefore, the discussion in section 2.3 is also applicable to the present situation.

Furthermore, when the controlled system described by Eq. (2.36) is subjected to additive Gaussian white noises and the output observation is the state variables contaminated by additive Gaussian white noises, the optimal stochastic control problem which minimizes the expectation of Eq. (2.37) can be treated from the performance loss point of view. In this case, we can combine the results in sections 2.3 and 2.5. A related work is presented by Meier & Anderson[35].

## 2.6 Concluding Remarks

The problem has been considered of the performance loss in the sequential estimation due to incorrect noise covariance matrices. We have derived the optimal, calculated, and actual covariance matrices, and the three difference equations are derived to express the evolution of error matrices. Numerical examples for a first order system clearly show that the Wiener-Kalman filter (or the Bayesian estimation method) is rather insensitive to variations of noise variances.

The remainder of the chapter is devoted to the investigation of the mutual relations between the estimation problem and the noise free regulator problem from the viewpoint of performance loss. We have demonstrated that there still exists the duality principle between the two problems in the sense of performance loss.

## CHAPTER 3

### PERFORMANCE LOSS FOR SEQUENTIAL STATE ESTIMATION WITH ERRORS IN SYSTEM PARAMETERS

#### 3.1 Introduction

In Chapter 1, we have considered the performance loss due to errors in noise covariance matrices; furthermore, the related problem of duality between the estimation and the noise free regulator problem is treated from performance loss point of view.

In the first part of this chapter, the performance loss, which is caused by errors in both plant and observation models, is evaluated by extending the method developed in the previous chapter.

In the second part, a different method of determining a suboptimal gain of the Wiener-Kalman filter is presented when the elements of both the transition and observation matrices randomly fluctuate around their

nominal values. To this end, the result of the first part is utilized. The suboptimal gain obtained is the optimal one to the linear filter designed by using the available information.

### 3.2 Statement of Problem

Let us consider a dynamical system described by the difference equation of the form

$$x(k+1) = [\Phi(k) + \delta\Phi(k)]x(k) + w(k), \quad (3.1)$$

where

$x(k)$  is an  $n$ -dimensional state vector,

$w(k)$  is an  $n$ -dimensional vector denoting the random disturbance,

$\Phi(k)$  is an  $n \times n$  state transition matrix, nominal value,

$\delta\Phi(k)$  is an  $n \times n$  matrix denoting the deviation of transition matrix from the nominal one.

Let the output observation be given by

$$z(k) = [H(k) + \delta H(k)]x(k) + v(k), \quad (3.2)$$

where

$z(k)$  is a  $p$ -dimensional observation vector,

$v(k)$  is a  $p$ -dimensional vector denoting the observation noise,

$H(k)$  is a  $p \times n$  observation matrix, nominal value,

$\delta H(k)$  is a  $p \times n$  matrix denoting the deviation of observation matrix from the nominal one.

It is assumed that the  $w(k)$  and  $v(k)$  are mutually independent Gaussian white noises with means zero and covariance matrices  $Q(k)$  and  $R(k)$ , respectively.

Now suppose that

$$\hat{\mathbf{x}}(k+1) = \Phi(k)\hat{\mathbf{x}}(k) + \mathbf{w}(k) \quad (3.3)$$

and

$$\hat{\mathbf{z}}(k) = \mathbf{H}(k)\hat{\mathbf{x}}(k) + \mathbf{v}(k) \quad (3.4)$$

be the erroneous mathematical model of the actual system described by Eqs. (3.1) and (3.2), where  $\hat{\mathbf{x}}(k)$  and  $\hat{\mathbf{z}}(k)$  denote the state and the output observation obtained by letting  $\delta\Phi(k) \equiv 0$  and  $\delta\mathbf{H}(k) \equiv 0$  in Eqs. (3.1) and (3.2), respectively.

If the actual values of  $\delta\Phi(k)$  and  $\delta\mathbf{H}(k)$  are not known, we can not construct the optimal filter for the actual system. A suboptimal filter for the actual system, however, can be obtained by using the model, Eqs. (3.3) and (3.4), and is expressed by the following recursive relations[26].

(i) The suboptimal estimates of the state are successively produced by

$$\mathbf{x}^*(k+1|k) = \Phi(k)\mathbf{x}^*(k|k-1) + \Gamma_c(k)[\mathbf{z}(k) - \mathbf{H}(k)\mathbf{x}^*(k|k-1)], \quad (3.5)$$

where it should be noted that the input to the suboptimal filter is  $\mathbf{z}(k)$  given by Eq. (3.2) and not  $\hat{\mathbf{z}}(k)$  given by Eq. (3.4).

(ii) The suboptimal gain matrix is defined by

$$\Gamma_c(k) = \Phi(k)P_c(k)H'(k)[H(k)P_c(k)H'(k) + R(k)]^{-1}. \quad (3.6)$$

(iii) The  $P_c(k)$  is the solution of the matrix Riccati equation

$$\begin{aligned} P_c(k+1) = & [\Phi(k) - \Gamma_c(k)H(k)]P_c(k)[\Phi(k) - \Gamma_c(k)H(k)]' \\ & + \Gamma_c(k)R(k)\Gamma_c'(k) + Q(k). \end{aligned} \quad (3.7)$$

The initial conditions to Eqs. (3.5) and (3.7) are given by  $\mathbf{x}^*(0|-1)$

$= 0$  and  $P_c(0) = E\{x(0)x'(0)\}$ , respectively.

In the next section, we consider the performance loss of the sub-optimal filter defined by Eqs. (3.5), (3.6) and (3.7). It is assumed for simplicity that the noise covariance matrices  $Q(k)$  and  $R(k)$  as well as the initial conditions are correct. It is of course possible to extend the result to a more general situation by combining the method of Chapter 2 with that of this chapter.

### 3.3 Derivation of Actual Error Covariance Matrix

The  $P_c(k)$  in Eq. (3.7) is the calculated error covariance matrix. On the other hand, the actual error covariance matrix is evaluated as follows. Let  $e(k)$  be the actual estimation error, i.e.,

$$e(k) = x(k) - x^*(k|k-1), \quad (3.8)$$

then, by using Eqs. (3.1) and (3.5), the evolution of  $e(k)$  can be expressed as

$$\begin{aligned} e(k+1) = & [\Phi(k) - \Gamma_c(k)H(k)]e(k) + [\delta\Phi(k) - \Gamma_c(k)\delta H(k)]x(k) \\ & - \Gamma_c(k)v(k) + w(k). \end{aligned} \quad (3.9)$$

This equation is different from Eq. (2.17) in Chapter 2; the second term, which is proportional to the actual state  $x(k)$ , does not appear in Eq. (2.17). For the further development, we must combine Eq. (3.9) with Eq. (3.1), obtaining the  $2n$  difference equations of the form

$$\begin{bmatrix} e(k+1) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} \Phi(k) - \Gamma_c(k)H(k) & \delta\Phi(k) - \Gamma_c(k)\delta H(k) \\ 0 & \Phi(k) + \delta\Phi(k) \end{bmatrix} \begin{bmatrix} e(k) \\ x(k) \end{bmatrix}$$



$$- \begin{bmatrix} \Gamma_c(k) \\ 0 \end{bmatrix} v(k) + \begin{bmatrix} I \\ I \end{bmatrix} w(k) \quad (3.10)$$

Now define

$$\Pi(k) = E\{e(k)e'(k)\}, \quad (3.11)_1$$

$$\Psi(k) = E\{e(k)x'(k)\}, \quad (3.11)_2$$

$$\Sigma(k) = E\{x(k)x'(k)\}. \quad (3.11)_3$$

Postmultiplying Eq. (3.10) by its transpose, and taking the expectation of both sides with respect to  $w(k)$  and  $v(k)$ , we have the difference equations describing the evolution of  $\Pi$ ,  $\Psi$  and  $\Sigma$  as follows.

$$\begin{aligned} \Pi(k+1) = & [\Phi(k) - \Gamma_c(k)H(k)]\Pi(k)[\Phi(k) - \Gamma_c(k)H(k)]' \\ & + [\Phi(k) - \Gamma_c(k)H(k)]\Psi(k)[\delta\Phi(k) - \Gamma_c(k)\delta H(k)]' \\ & + [\delta\Phi(k) - \Gamma_c(k)\delta H(k)]\Psi'(k)[\Phi(k) - \Gamma_c(k)H(k)]' \\ & + [\delta\Phi(k) - \Gamma_c(k)\delta H(k)]\Sigma(k)[\delta\Phi(k) - \Gamma_c(k)\delta H(k)]' \\ & + \Gamma_c(k)R(k)\Gamma_c'(k) + Q(k), \end{aligned} \quad (3.12)$$

$$\begin{aligned} \Psi(k+1) = & [\Phi(k) - \Gamma_c(k)H(k)]\Psi(k)[\Phi(k) + \delta\Phi(k)]' \\ & + [\delta\Phi(k) - \Gamma_c(k)\delta H(k)]\Sigma(k)[\Phi(k) + \delta\Phi(k)]' + Q(k), \end{aligned} \quad (3.13)$$

$$\Sigma(k+1) = [\Phi(k) + \delta\Phi(k)]\Sigma(k)[\Phi(k) + \delta\Phi(k)]' + Q(k), \quad (3.14)$$

where the fact that both the  $w(k)$  and  $v(k)$  are mutually independent Gaussian white has been used. Eq. (3.12) shows the evolution of the actual error covariance matrix, and is related to  $\Psi(k)$  and  $\Sigma(k)$  due to the existence of the variations  $\delta\Phi(k)$  and  $\delta H(k)$ . Moreover, Eq. (3.12)

is rather different from Eq. (2.19); that is, in the previous result the  $\Psi(k)$  and  $\Sigma(k)$  were not necessary for the evaluation of the actual error covariance matrix.

On the other hand, the optimal covariance matrix  $P_o(k)$  associated with the Wiener-Kalman filter for the actual dynamical and observation system, which is not realizable in the present case, is given by

$$P_o(k+1) = [\Phi_1(k) - \Gamma_o(k)H_1(k)]P_o(k)[\Phi_1(k) - \Gamma_o(k)H_1(k)]' + \Gamma_o(k)R(k)\Gamma_o'(k) + Q(k), \quad (3.15)$$

where

$$\Gamma_o(k) = \Phi_1(k)P_o(k)H_1'(k)[H_1(k)P_o(k)H_1'(k) + R(k)]^{-1} \quad (3.16)$$

and

$$\Phi_1(k) = \Phi(k) + \delta\Phi(k), \quad (3.17)_1$$

$$H_1(k) = H(k) + \delta H(k). \quad (3.17)_2$$

Therefore, the performance loss of the suboptimal filter can be evaluated by

$$E_{ao}(k) = \Pi(k) - P_o(k). \quad (3.18)$$

The  $E_{ca}(k)$  and  $E_{co}(k)$  can also be evaluated from Eqs. (3.7), (3.12) and (3.15) as their mutual differences. However, the interesting assertions such as given in section 2.3 do not directly follow, so that we shall not pursue any more. To make the above discussion more clear, we shall give some numerical results for a first order system in the next section.

In Chapter 2, we have also proved that when the covariance matrices of system and observation noises are incorrectly identified, there still

exists the duality principle between the estimation and the noise free regulator problem with incorrect weighting matrices in the criterion function. In the present case, however, the related duality principle does not hold in the sense of performance loss.

In fact, let the criterion function of the noise free regulator problem be

$$I_N = \sum_{j=0}^{N-1} \{ \|x(j+1)\|_{\Lambda(j+1)}^2 + \|u(j)\|_{\Theta(j)}^2 \} \quad (3.19)$$

subject to the dynamical constraint

$$x(k+1) = F_1(k)x(k) + G_1(k)u(k). \quad (3.20)$$

If we use incorrect matrices  $F(k)$  and  $G(k)$  rather than the correct  $F_1(k)$  and  $G_1(k)$ , respectively, then from Eq. (2.49) the control function becomes

$$u_c(k) = -B_c(N-k)x(k), \quad (3.21)$$

where, in this case,

$$B_c(N-k) = [L_{GG}^c(N-k+1) + R(k)]^{-1} L_{GF}^c(N-k+1). \quad (3.22)$$

Referring to Eqs. (2.48) and (3.7), we observe that the  $S_c(N-k)$  and  $P_c(k+1)$  are mutually dual. On the other hand, from Eq. (2.54), the matrix Riccati equation defining the actual performance becomes

$$\begin{aligned} S_a(N-k) = & [F_1(k) - G_1(k)B_c(N-k)]' S_a(N-k+1) [F_1(k) - G_1(k)B_c(N-k)] \\ & + B_c'(N-k)\Theta(k)B_c(N-k) + \Lambda(k). \end{aligned} \quad (3.23)$$

Therefore, there does not exist any correspondences between Eqs. (3.12) and (3.23); that is, there is no duality relation in the sense of performance loss.

### 3.4 Qualitative Aspects of Performance Loss for a First Order System

Some numerical results for the performance loss problem discussed in section 3.3 are presented. Consider a first order system described by

$$x(k+1) = (a + \delta a)x(k) + w(k) \quad (3.24)$$

and

$$z(k) = (h + \delta h)x(k) + v(k), \quad (3.25)$$

where  $w(k)$  and  $v(k)$  are mutually independent Gaussian white noises with means zero and variances  $q$  and  $r$ , respectively. Moreover, it is assumed that all the parameters  $a$ ,  $\delta a$ ,  $h$ ,  $\delta h$  are constant.

Applying the previous result, we obtain the following equations describing the actual estimation error.

$$\begin{aligned} \Pi(k+1) = & [a - h\Gamma_c(k)]^2 \Pi(k) + 2[a - h\Gamma_c(k)][\delta a - \delta h\Gamma_c(k)]\Psi(k) \\ & + [\delta a - \delta h\Gamma_c(k)]^2 \Sigma(k) + r\Gamma_c^2(k) + q, \end{aligned} \quad (3.26)$$

$$\begin{aligned} \Psi(k+1) = & [a - h\Gamma_c(k)][a + \delta a]\Psi(k) \\ & + [\delta a - \delta h\Gamma_c(k)][a + \delta a]\Sigma(k) + q, \end{aligned} \quad (3.27)$$

$$\Sigma(k+1) = [a + \delta a]^2 \Sigma(k) + q, \quad (3.28)$$

where

$$\Gamma_c(k) = ahP_c(k)/[r + h^2P_c(k)] \quad (3.29)$$

and

$$P_c(k+1) = a^2rP_c(k)/[r + h^2P_c(k)] + q. \quad (3.30)$$

While the estimation error associated with the optimal Wiener-Kalman filter for the actual system becomes, from Eq. (3.15),

$$P_o(k+1) = a_1^2 r P_o(k) / [r + h_1^2 P_o(k)] + q, \quad (3.31)$$

where the optimal gain is

$$\Gamma_o(k) = a_1 h_1 P_o(k) / [r + h_1^2 P_o(k)], \quad (3.32)$$

where  $a_1 = a + \delta a$  and  $h_1 = h + \delta h$ .

Numerical values used in the computation are as follows.

$$a_1 = 0.9, \quad h_1 = 1.0, \quad q = r = 1.0.$$

Figs. 3.1 and 3.2 show the stationary performance losses

$$E_{ao} = E_{ao}(k) \big|_{k \rightarrow \infty},$$

due to variations  $\delta a$  and  $\delta h$  around their nominal values, respectively.

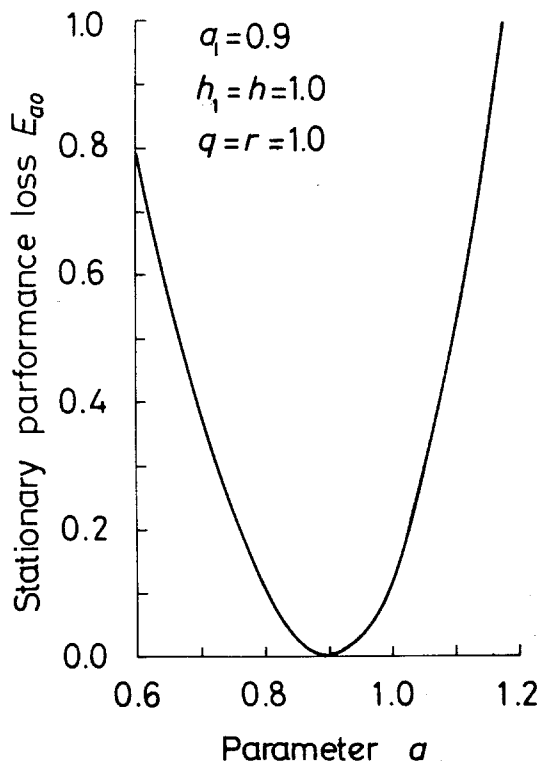


Fig. 3.1

Stationary performance loss due to misidentification of system parameter  $a_1$

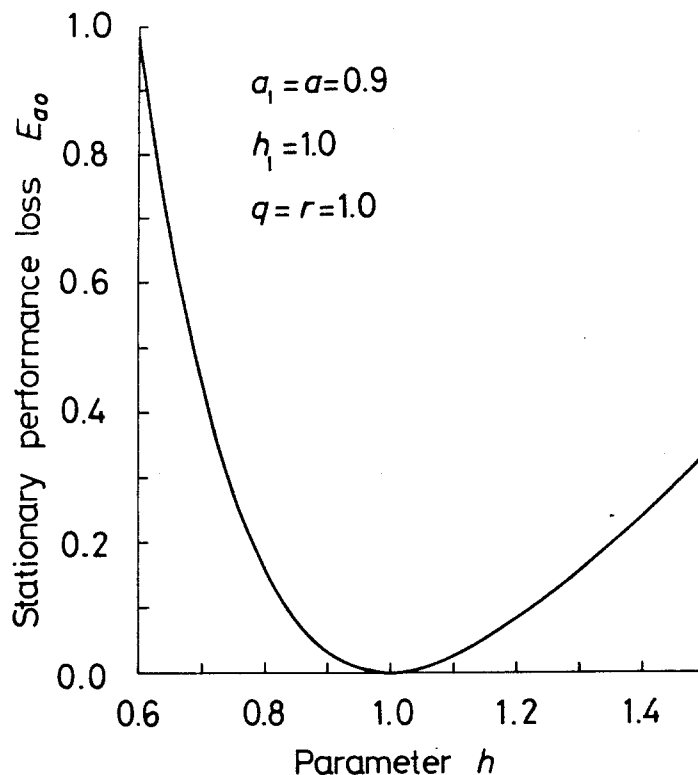


Fig. 3.2 Stationary performance loss due to misidentification of observation parameter  $h_1$

### 3.5 Determination of a Suboptimal Gain for Linear Systems with Random Parameters

Recently, Neal[36] has considered a method of determining a sub-optimal gain of the Wiener-Kalman filter in the presence of errors in assumed plant dynamics. In this section, we present a different method of determining a suboptimal gain for the discrete-time Wiener-Kalman filter when the elements of both the transition and observation matrices are randomly time-varying. More precisely, it is assumed that the elements of  $n \times n$  matrices  $\delta\Phi(k)$ , say  $\phi_{ij}$ , are Gaussian with

$$E\{\phi_{ij}\} = 0 \quad (3.33)_1$$

and

$$E\{\phi_{im}\phi_{jl}\} = \alpha_{im}^2 \delta_{ij} \delta_{lm}, \quad (3.33)_2$$

and that the elements of  $p \times n$  matrix  $H(k)$ , say  $h_{ij}$ , are also Gaussian with

$$E\{h_{ij}\} = 0 \quad (3.34)$$

and

$$E\{h_{im}h_{jl}\} = \beta_{im}^2 \delta_{ij} \delta_{lm}, \quad (3.34)_2$$

where both the  $\alpha_{im}^2$  and  $\beta_{im}^2$  are variances.

To make the present problem more clear, let us consider the first order system treated in the previous example. Furthermore, it is assumed here that  $\delta a(k)$  and  $\delta h(k)$  are mutually independent Gaussian white noises with means zero and variances  $\sigma_a^2$  and  $\sigma_h^2$ , respectively. The a posteriori probability density function of  $x(k+1)$  conditioned on  $Z_k = \{z(0), \dots, z(k)\}$  is expressed as

$$\begin{aligned} p(x(k+1)|Z_k) &= \int_{-\infty}^{\infty} p(x(k+1)|x(k))p(x(k)|Z_k)dx(k) \\ &= C_1 \int_{-\infty}^{\infty} p(x(k+1)|x(k))p(z(k)|x(k))p(x(k)|Z_{k-1})dx(k), \end{aligned} \quad (3.35)$$

where  $C_1$  is a normalizing factor. By using Eqs. (3.24) and (3.25), it follows that

$$p(x(k+1)|Z_k) = C_1 \int_{-\infty}^{\infty} \exp\left\{-\frac{[x(k+1) - ax(k)]^2}{2[q + \sigma_a^2 x^2(k)]}\right\} -$$

$$- \frac{[z(k) - hx(k)]^2}{2[r + \sigma_h^2 x^2(k)]} \} p(x(k)|Z_{k-1}) dx(k). \quad (3.36)^\dagger$$

The integration of the right hand side of this equation is almost impossible except for the case where  $\sigma_a^2 = \sigma_h^2 = 0$  and the a priori probability density function  $p(x(0)|Z_{-1})$  of the initial state is Gaussian. Therefore, at the present stage, it is almost impossible to find the optimal solution of this problem, and the filter equation, if this could be obtained, will be of complex nonlinear form.

Abandoning the desire to obtain the optimal filter, let us now consider to find the best linear filter. Suppose that the best linear filter has the form

$$x_s^*(k+1|k) = \Phi(k)x_s^*(k|k-1) + \Gamma_s(k)[z(k) - H(k)x_s^*(k|k-1)], \quad (3.37)$$

where  $\Gamma_s(k)$  remains to be determined so as to minimize the variance of estimation error. From Eqs. (3.1) and (3.37), we have

$$\begin{aligned} e_s(k+1) &= [\Phi(k) - \Gamma_s(k)H(k)]e_s(k) + [\delta\Phi(k) - \Gamma_s(k)\delta H(k)]x(k) \\ &\quad - \Gamma_s(k)v(k) + w(k), \end{aligned} \quad (3.38)$$

where  $e_s(k) = x(k) - x_s^*(k|k-1)$ . This equation is the same form as Eq. (3.9); so that the result in section 3.3 is applicable to the evaluation of the error covariance matrix of the estimate  $x_s^*(k|k-1)$ .

The careful study of Eqs. (3.12), (3.13) and (3.14) shows that, if we take the expectation of these equations with respect to  $\delta\Phi(k)$  and

---

† The derivation of Eq. (3.36) is omitted. See for example Chapter 4; there will be found the detail of the derivation of the a posteriori probability density function based on the Bayes theorem[18].



$\delta H(k)$  by replacing  $\Gamma_c(k)$  by  $\Gamma_s(k)$ , then the difference equation is obtained, which describes the evolution of the covariance matrix when the elements of  $\delta\Phi(k)$  and  $\delta H(k)$  are randomly time-varying and the filter is assumed to be linear.

By making use of Eqs. (3.33)<sub>1</sub> and (3.34)<sub>1</sub>, it follows from Eqs. (3.12), (3.13) and (3.14) that

$$\begin{aligned}\tilde{\Pi}(k+1) &= [\Phi(k) - \Gamma_s(k)H(k)]\tilde{\Pi}(k)[\Phi(k) - \Gamma_s(k)H(k)]' \\ &\quad + \Gamma_s(k)[E\{\delta H(k)\Sigma(k)\delta H'(k)\} + R(k)]\Gamma_s'(k) \\ &\quad + E\{\delta\Phi(k)\Sigma(k)\delta\Phi'(k)\} + Q(k),\end{aligned}\quad (3.39)$$

$$\begin{aligned}\tilde{\Psi}(k+1) &= [\Phi(k) - \Gamma_s(k)H(k)]\tilde{\Psi}(k)\Gamma'(k) \\ &\quad + E\{\delta\Phi(k)\Sigma(k)\delta\Phi'(k)\} + Q(k),\end{aligned}\quad (3.40)$$

$$\tilde{\Sigma}(k+1) = \Phi(k)\Sigma(k)\Phi'(k) + E\{\delta\Phi(k)\Sigma(k)\delta\Phi'(k)\} + Q(k),\quad (3.41)$$

where the tilde( $\sim$ ) denotes the expectation with respect to  $\delta\Phi$  and  $\delta H$ .

The expectation  $E\{\delta\Phi\Sigma(k)\delta\Phi'\}$  can easily be computed as follows by using Eqs. (3.33)<sub>2</sub> and (3.34)<sub>2</sub>. The  $(i,j)$  element of the expectation becomes

$$\begin{aligned}E\{\delta\Phi\Sigma(k)\delta\Phi'\}_{ij} &= E\left\{\sum_{m=1}^n \sum_{\ell=1}^n \phi_{i\ell} \phi_{jm} \Sigma_{\ell m}(k)\right\} \\ &= \delta_{ij} \sum_{\ell=1}^n \alpha_{i\ell}^2 \Sigma_{\ell\ell}(k).\end{aligned}\quad (3.42)$$

Therefore, this is an  $n \times n$  diagonal matrix with element

$$\sum_{\ell=1}^n \alpha_{i\ell}^2 \Sigma_{\ell\ell}(k), \quad i = 1, 2, \dots, n. \quad (3.43)$$

Similarly, the expectation  $E\{\delta H \Sigma(k) \delta H'\}$  becomes a  $p \times p$  diagonal matrix with element

$$\sum_{\ell=1}^n \beta_{j\ell}^2 \gamma_{\ell\ell}(k), \quad j = 1, 2, \dots, p. \quad (3.44)$$

Next, we consider the problem of determining the gain  $\Gamma_s(k)$ . We determine the gain so as to minimize the  $\hat{\Sigma}(k+1)$ , which is the covariance matrix of estimation error and is given by Eq. (3.39). This method is motivated by the fact that the optimal gain of the Wiener-Kalman filter minimizes the covariance of estimation error, as was shown in Eq. (2.10). Referring to Eq. (2.11),  $\Gamma_s(k)$  is found to be

$$\begin{aligned} \Gamma_s(k) &= \Phi(k) \hat{\Sigma}(k) H'(k) \\ &\times [H(k) \hat{\Sigma}(k) H'(k) + E\{\delta H(k) \Sigma(k) \delta H'(k)\} + R(k)]^{-1}. \end{aligned} \quad (3.45)$$

This equation is the main result of this section.

This solution can be also derived from Eq. (3.36) by assuming that

$$\sigma_a^2 x^2(k) \approx \sigma_a^2 E\{x^2(k)\} \quad (3.46)_1$$

and

$$\sigma_h^2 x^2(k) \approx \sigma_h^2 E\{x^2(k)\}. \quad (3.46)_2$$

In fact, if we use the above assumption, and if we assume that the a priori probability density function  $p(x(0)|Z_{-1})$  is Gaussian, then the integration of Eq. (3.36) is straightforward; and we have the usual Wiener-Kalman filter with the replacements:

$$q \rightarrow q + \sigma_a^2 E\{x^2(k)\} \quad (3.47)_1$$

and

$$r \rightarrow r + \sigma_h^2 E\{x^2(k)\}. \quad (3.47)_2$$

This is an interpretation of the result, Eq. (3.45), from the Bayesian point of view.

### 3.6 Digital Simulation Studies

The suboptimal gain determined above is computed and compared with the optimal and calculated gains for the same system treated in section 3.4, i.e.,

$$x(k+1) = [a + \delta a(k)]x(k) + w(k) \quad (3.48)$$

and

$$z(k) = [h + \delta h(k)]x(k) + v(k), \quad (3.49)$$

where  $\delta a(k)$  and  $\delta h(k)$  are mutually independent Gaussian white noises with means zero and variances  $\sigma_a^2$  and  $\sigma_h^2$ , respectively.

Taking the expectation of Eqs. (3.26), (3.27) and (3.28), it follows that

$$\begin{aligned} \hat{x}(k+1) &= [a - h\Gamma_s(k)]^2 \hat{x}(k) + [\sigma_a^2 + \sigma_h^2 \Gamma_s^2(k)] \hat{x}(k) \\ &\quad + r\Gamma_s^2(k) + q, \end{aligned} \quad (3.50)$$

$$\hat{y}(k+1) = a[a - h\Gamma_s(k)]\hat{y}(k) + \sigma_a^2 \Gamma_s(k) + q, \quad (3.51)$$

$$\hat{z}(k+1) = [a^2 + \sigma_a^2] \hat{z}(k) + q. \quad (3.52)$$

Therefore, the suboptimal gain  $\Gamma_s(k)$ , a scalar version of Eq. (3.45), becomes

$$\Gamma_s(k) = ah\hat{x}(k)/[r + \sigma_h^2 \hat{z}(k) + h^2 \hat{x}(k)]. \quad (3.53)$$

Now we have three different estimators whose gains are  $\Gamma_c(k)$ ,  $\Gamma_s(k)$  and  $\Gamma_o(k)$ , respectively, i.e.,

$$x_c^*(k+1|k) = ax_c^*(k|k-1) + \Gamma_c(k)[z(k) - hx_c^*(k|k-1)], \quad (3.54)$$

$$x_s^*(k+1|k) = ax_s^*(k|k-1) + \Gamma_s(k)[z(k) - hx_s^*(k|k-1)], \quad (3.55)$$

$$\hat{x}(k+1|k) = a_1 \hat{x}(k|k-1) + \Gamma_o(k)[z(k) - h_1 \hat{x}(k|k-1)], \quad (3.56)$$

where  $a_1 = a + \delta a(k)$  and  $h_1 = h + \delta h(k)$ , and where

$$\Gamma_c(k) = ahP_c(k)/[r + h^2P_c(k)], \quad (3.57)$$

$$\Gamma_o(k) = a_1 h_1 P_o(k)/[r + h_1^2 P_o(k)], \quad (3.58)$$

and  $P_c(k)$  and  $P_o(k)$  are respectively given by Eqs. (3.30) and (3.31).

Fig. 3.3 displays the comparison of the behavior of the three different gains; the following numerical values are used.

$$a = 0.9, \quad h = 1.0, \quad q = 0.01, \quad r = 0.09, \quad \sigma_a^2 = 0.09,$$

$$\sigma_h^2 = 0.0, \quad P_c(0) = P_o(0) = \hat{\Pi}(0) = \hat{\Sigma}(0) = 4.0.$$

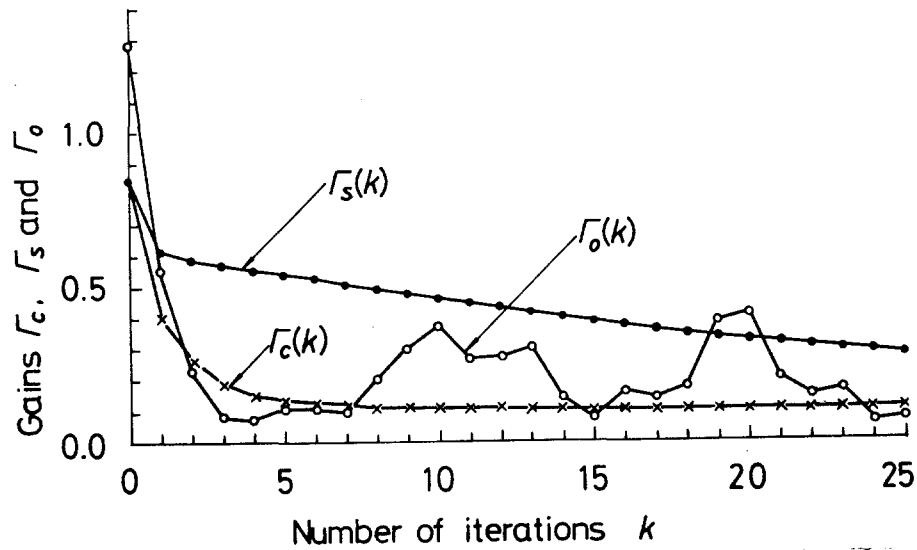


Fig. 3.3 Comparison of three different gains  $\Gamma_o(k)$ ,  $\Gamma_s(k)$  and  $\Gamma_c(k)$

The optimal gain  $\Gamma_o(k)$  changes according to the random variation  $\delta a(k)$ , and shows the different behavior for each different trials. The  $\Gamma_s(k)$  shows rather different behavior from the  $\Gamma_c(k)$ . This is because, though the calculated gain  $\Gamma_c(k)$  is completely determined by the assumed model with  $\delta a(k) \equiv 0$ , the suboptimal gain  $\Gamma_s(k)$  is affected by the actual variance  $\hat{\Sigma}(k)$  of the state  $x(k)$  through the variance  $\sigma_a^2$  of  $\delta a(k)$ .

Figs. 3.4 and 3.5 display those performances of the three different estimators defined above which are evaluated by a digital computer with pseudorandom numbers. In these figures,

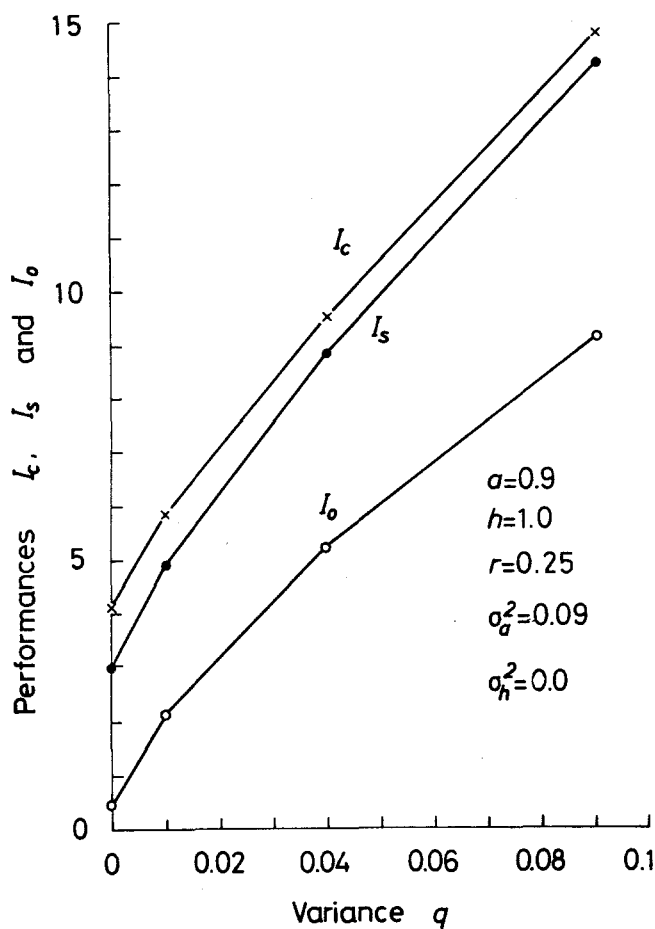


Fig. 3.4

Experimental evaluation of the filters performances  $I_o$ ,  $I_s$  and  $I_c$

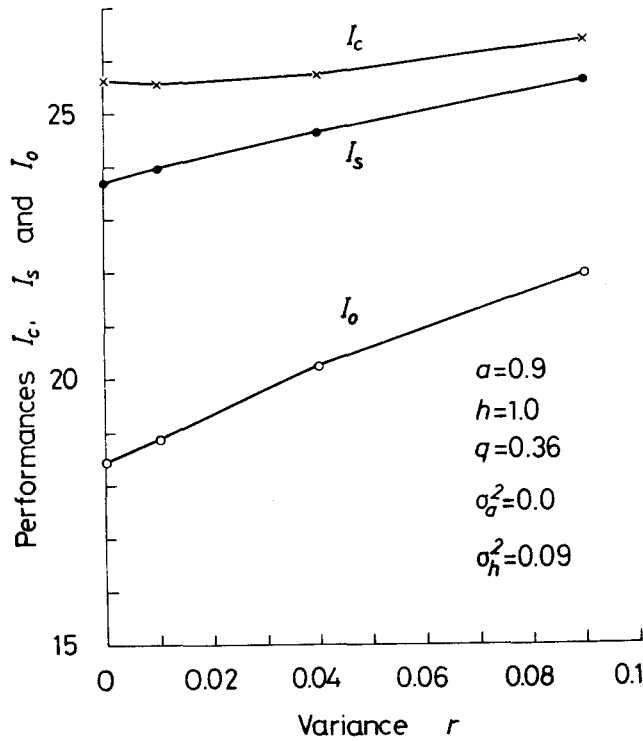


Fig. 3.5

Experimental evaluation of  
the filter performances  
 $I_o$ ,  $I_s$  and  $I_c$

$$I_o = E' \left\{ \sum_{k=1}^{50} [x(k) - \hat{x}(k|k-1)]^2 \right\} \quad (3.59)$$

$$I_s = E' \left\{ \sum_{k=1}^{50} [x(k) - \hat{x}_s^*(k|k-1)]^2 \right\} \quad (3.60)$$

and

$$I_c = E' \left\{ \sum_{k=1}^{50} [x(k) - \hat{x}_c^*(k|k-1)]^2 \right\}, \quad (3.61)$$

where  $x(k)$  is the actual state of the dynamical system defined by Eq. (3.48), and  $E'$  indicates the average of a hundred separate trials.

From Figs. 3.4 and 3.5, it is seen that the use of the  $\Gamma_s(k)$  improves the performance of the filter over the  $\Gamma_c(k)$ . The rate of the improvement is large on the range that the variances  $\sigma_a^2$  and  $\sigma_h^2$  dominate the variances of system and observation noises, respectively.

### 3.7 Concluding Remarks

We have considered, firstly, the performance loss of the discrete-time Wiener-Kalman filter designed on the basis of the model with errors in system and observation matrices. We have derived the difference equation which describes the evolution of the actual error covariance matrix. Moreover, it is found that the variance of the actual state of the system influences the actual error covariance matrix of the sub-optimal filter; this fact was not observed in Chapter 2 where the performance loss due to errors in noise covariance matrices are treated. On the contrary to the result in section 2.5 (Chapter 2), we can not find dual relations between the estimation problem and the noise free regulator problem from the viewpoint of performance loss.

In the second part, we have developed a method of determining a sub-optimal gain when the elements of the transition and observation matrices are randomly time-varying. The result of the first part is utilized to determine a suboptimal gain, which is optimal among linear filters of the form given by Eq. (3.37). Digital simulation studies show that the suboptimal gain  $\Gamma_s(k)$  is superior to  $\Gamma_c(k)$  when the parameters of the transition and observation matrices are randomly time-varying.

## CHAPTER 4

### ESTIMATION OF STATE VARIABLES FOR NONLINEAR DYNAMICAL SYSTEMS

#### 4.1 Introduction

The problem of nonlinear filtering was first treated by Zadeh[57], and later by Balakrishnan[3], etc. Stratonovich[50] has suggested that the solution of the optimal nonlinear filtering follows from the a posteriori probability density function of the signal process conditioned on the observed data. The approach along this line is due to Stratonovich [50], Kushner[31], Wonham[55], and Bucy[8]; they derived the stochastic differential equation satisfied by the conditional probability density function of the state of nonlinear dynamical systems. Bass et al.[4] extended the above result and derived an approximation method of estimating state variables of nonlinear systems; Cox[9] applied the method



of dynamic programming to the nonlinear sequential estimation problem; Sorenson & Stubberud[48] approximately solved a nonlinear filtering problem by using Gaussian approximation to the a posteriori probability density function.

In this chapter, the estimation of state variables for discrete-time nonlinear dynamical systems is considered. The main purpose is to derive an approximation technique which is applicable to nonlinear noisy observation problem. The principal approach is based on the statistical linearization technique under the assumption that the a posteriori probability density function of the state is Gaussian[40].

#### 4.2 Statement of Problem

We shall begin with the general formulation of the problem of estimating state variables for discrete-time nonlinear dynamical systems subjected to a Gaussian white random input. Consider a process described by a vector difference equation of the form

$$x(k+1) = f(x(k), k) + w(k), \quad (4.1)$$

where

$x(k)$  is an  $n$ -dimensional state vector,

$w(k)$  is an  $n$ -dimensional vector denoting the random input,

$f(\cdot, \cdot)$  is an  $n$ -dimensional vector-valued nonlinear function.

It should be noted that in Eq. (4.1) if there exist any known forcing functions, they can be included in the nonlinear term  $f(x(k), k)$  which is explicitly dependent on time  $k$ . Let the output observation be nonlinear combination of the state variables corrupted by additive noise,

that is,

$$z(k) = h(x(k), k) + v(k), \quad (4.2)$$

where

$z(k)$  is a  $p$ -dimensional vector denoting the output observation,

$v(k)$  is a  $p$ -dimensional vector denoting the observation noise,

$h(\cdot, \cdot)$  is a  $p$ -dimensional vector-valued nonlinear function.

It is assumed that the processes  $w(k)$  and  $v(k)$  are mutually independent Gaussian white noises with means zero and covariance matrices  $Q(k)$  and  $R(k)$ , respectively, and that the initial state  $x(0)$  is Gaussian and is independent of the  $w(k)$  and  $v(k)$ . Moreover, it is assumed that the covariance matrices  $Q(k)$  and  $R(k)$  are both nonsingular. However, this assumption will be removed in section 4.5. Fig. 4.1 shows the schematic representation of the nonlinear system to be considered.

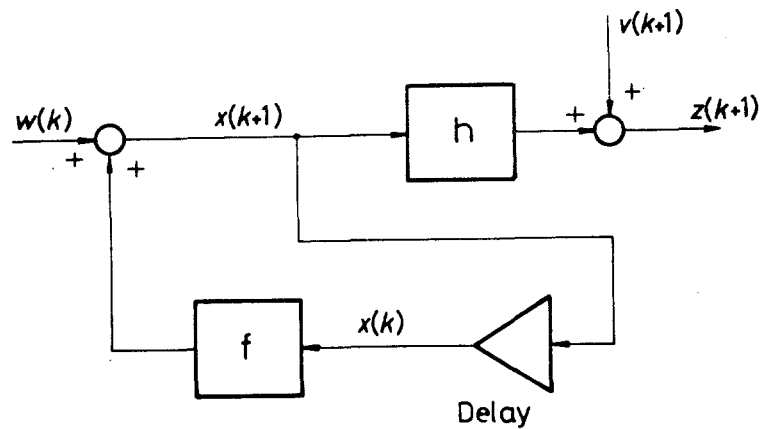


Fig. 4.1 Schematic representation of the system to be considered

The nonlinear state estimation problem considered is to find the optimal estimate  $\hat{x}(k|k)$  of the current state  $x(k)$ , on the basis of the observation up to time  $k$ , so as to minimize the conditional expectation of the quadratic loss function relative to  $Z_k$ , i.e.,

$$E\{\|x(k) - \hat{x}(k|k)\|^2 | Z_k\}, \quad (4.3)$$

where  $Z_k = \{z(1), \dots, z(k)\}$ . As mentioned in Chapter 1, the optimal estimate is given by

$$\hat{x}(k|k) = E\{x(k) | Z_k\}. \quad (4.4)$$

It is clear from this equation that the important part of the state estimation problem is the evaluation of the a posteriori probability density function  $p(x(k) | Z_k)$  of the state conditioned on the observed data  $Z_k$ .

#### 4.3 A Posteriori Probability Density Function

By making use of the Bayes theorem[18], the joint a posteriori probability density function of the states  $X_k = \{x(0), \dots, x(k)\}$  conditioned on  $Z_k$  can be written as

$$p(X_k | Z_k) = \frac{p(X_k)p(Z_k | X_k)}{p(Z_k)}. \quad (4.5)$$

Since, in Eqs. (4.1) and (4.2), both the  $w(k)$  and  $v(k)$  are Gaussian white, both the  $x(k)$  and  $z(k)$  become Markov processes. Then, it follows that

$$p(X_k) = p(x(0)) \prod_{j=0}^{k-1} p(x(j+1) | x(j)) \quad (4.6)$$

and

$$p(Z_k | X_k) = \prod_{j=1}^k p(z(j) | x(j)), \quad (4.7)$$

where  $p(x(0))$  is the a priori probability density function of the initial state, and  $p(x(j+1) | x(j))$  and  $p(z(j) | x(j))$  are conditional probability density functions. It is easily seen from Eqs. (4.1) and (4.2) that  $p(x(j+1) | x(j))$  is Gaussian with mean  $f(x(j), j)$  and covariance matrix  $Q(j)$  and that  $p(z(j) | x(j))$  is also Gaussian with mean  $h(x(j), j)$  and covariance matrix  $R(j)$ , i.e.,

$$p(x(j+1) | x(j)) = (2\pi)^{-n/2} |Q(j)|^{-1/2} \exp\left\{-\frac{1}{2} \|x(j+1) - f(x(j), j)\|_{Q^{-1}(j)}^2\right\} \quad (4.8)$$

and

$$p(z(j) | x(j)) = (2\pi)^{-p/2} |R(j)|^{-1/2} \exp\left\{-\frac{1}{2} \|z(j) - h(x(j), j)\|_{R^{-1}(j)}^2\right\}. \quad (4.9)$$

The assumption that both the covariance matrices  $Q(j)$  and  $R(j)$  are non-singular is necessary to derive Eqs. (4.8) and (4.9).

Substituting Eqs. (4.8) and (4.9) into Eqs. (4.6) and (4.7), respectively, we have

$$p(X_k) = C_1 \exp\left\{-\frac{1}{2} \sum_{j=0}^{k-1} \|x(j+1) - f(x(j), j)\|_{Q^{-1}(j)}^2 - \frac{1}{2} \|x(0)\|_{M^{-1}(0)}^2\right\} \quad (4.10)$$

and

$$p(Z_k | X_k) = C_2 \exp\left\{-\frac{1}{2} \sum_{j=1}^k \|z(j) - h(x(j), j)\|_{R^{-1}(j)}^2\right\}, \quad (4.11)$$

where  $C_1$  and  $C_2$  are normalizing factors. It is assumed that the a

priori probability density function  $p(x(0))$  is Gaussian with mean zero and covariance matrix  $M(0)$  which is positive definite. Substituting Eqs. (4.10) and (4.11) into Eq. (4.5) yields

$$p(X_k | Z_k) = C_3 \exp \left\{ -\frac{1}{2} \sum_{j=1}^k \|z(j) - h(x(j), j)\|_{R^{-1}(j)}^2 - \frac{1}{2} \sum_{j=0}^{k-1} \|x(j+1) - f(x(j), j)\|_{Q^{-1}(j)}^2 - \frac{1}{2} \|x(0)\|_{M^{-1}(0)}^2 \right\}, \quad (4.12)$$

where  $C_3$  is a normalizing constant independent of  $X_k$ , and is of no consequence here. Integrating Eq. (4.12) with respect to states  $x(0), \dots, x(k-1)$ , we finally obtain the a posteriori probability density function of the state  $x(0)$  conditioned on the observation  $Z_k$ :

$$p(x(k) | Z_k) = \int p(X_k | Z_k) dx(0) \dots dx(k-1). \quad (4.13)$$

Applying Eq. (4.13) to Eq. (4.4), we have the best estimate  $\hat{x}(k|k)$  of the current state  $x(k)$  in the sense that Eq. (4.3) is minimized. However, it is clear that the evaluation of the right hand side of Eq. (4.4) by using Eq. (4.13) is almost impossible. The difficulty apparently stems from the existence of the nonlinearities  $f(x(j), j)$  and  $h(x(j), j)$ . Furthermore, the above nonsequential estimation procedure is inconvenient because a repetition of the entire calculation is required whenever the available data is obtained.

Now consider the evolution of the a posteriori probability density function, that is, the relation between  $p(x(k) | Z_k)$  and  $p(x(k+1) | Z_{k+1})$ .

It follows from the Bayes theorem that

$$p(x(k+1) | Z_{k+1}) = \frac{p(x(k+1), z(k+1) | Z_k)}{p(z(k+1) | Z_k)}. \quad (4.14)$$

Since both the  $x(k)$  and  $z(k)$  are Markov processes, we have

$$p(x(k+1), z(k+1)|Z_k) = p(z(k+1)|x(k+1))p(x(k+1)|Z_k) \quad (4.15)_1$$

$$= \int p(z(k+1)|x(k+1))p(x(k+1)|x(k))p(x(k)|Z_k)dx(k). \quad (4.15)_2$$

By using Eqs. (4.8), (4.9) and (4.15)<sub>2</sub>, Eq. (4.14) becomes

$$p(x(k+1)|Z_{k+1}) = C_4 \int \exp\left\{-\frac{1}{2}\|z(k+1) - h(x(k+1), k+1)\|_{R^{-1}(k+1)}^2\right. \\ \left. - \frac{1}{2}\|x(k+1) - f(x(k), k)\|_{Q^{-1}(k)}^2\right\} p(x(k)|Z_k)dx(k), \quad (4.16)$$

where  $C_4$  is a normalizing factor independent of the state  $x_{k+1}$ . This is a difference-integral equation showing how the a posteriori probability density under consideration evolves with the increase of observations. This equation appears to be useful for finding the recursive relation for the best estimate. The existence of the nonlinearities, however, makes it difficult to perform the integration of the right hand side of Eq. (4.16). Hence, the direct application of Eq. (4.16) as well as Eq. (4.13) to the estimation problem is impossible from the computational point of view. Therefore, it is desirable to introduce an approximation technique in order to overcome the difficulty mentioned above.

#### 4.4 Approximation Technique for Nonlinear Systems

In this section, we consider an approximation technique to reduce the complexity caused by the nonlinearities  $f(x(k), k)$  and  $h(x(k), k)$ . The approximation technique presented here is based on the assumption that the a posteriori probability density function  $p(x(k)|Z_k)$  of the

state  $x(k)$  conditioned on the observed data  $Z_k$  is Gaussian with mean  $x^*(k|k)$  and covariance matrix  $M(k)$ , i.e.,

$$p(x(k)|Z_k) = (2\pi)^{-n/2} |M(k)|^{-1/2} \exp\left\{-\frac{1}{2} \|x(k) - x^*(k|k)\|_{M^{-1}(k)}^2\right\}. \quad (4.17)$$

for all  $k$ . It should be noted that  $x^*(k|k)$  denotes the approximate conditional mean, which amounts to the approximate estimate of the current state  $x(k)$  given observed data  $Z_k$ . In the sequel, the  $p(x(k)|Z_k)$  is to be understood as expressed in Eq. (4.17). The asterisk(\*) indicates that the  $x^*(k|k)$  is different from the best estimate  $\hat{x}(k|k)$  defined by Eq. (4.4). The  $M(k)$  is an  $n \times n$  covariance matrix defined by

$$M(k) = E\{[x(k) - x^*(k|k)][x(k) - x^*(k|k)]' | Z_k\}. \quad (4.18)$$

This concept is similar to that of the statistical linearization technique which is successfully applied to the analyses for nonlinear automatic control systems subjected to Gaussian random inputs[39]. Intuitively, the assumption expressed by Eq. (4.17) will be guaranteed approximately if the both nonlinearities are well behaved†. In order to derive the recursive relation between  $p(x(k)|Z_k)$  and  $p(x(k+1)|Z_{k+1})$ , it is necessary from the assumption that the both nonlinearities  $f(x(k), k)$  and  $h(x(k+1), k+1)$  are linearized with respect to the states  $x(k)$  and  $x(k+1)$ , respectively. If the both nonlinearities are linearized, the integration of the right hand side of Eq. (4.16) is straightforward. Thus the approximate relation which represents the evolution of the a posteriori probability density function can be obtained.

---

† An accuracy consideration will be presented in Chapter 5.

From the approximate relation obtained, we can readily construct recursive relations for the approximate estimate.

Consider the linearization of the nonlinear system described by Eq. (4.1). By using the concept of statistical linearization technique[39], we try to linearize Eq. (4.1) as

$$x(k+1) \approx F(k)[x(k) - x^*(k)] + f(x^*(k), k) + w(k), \quad (4.19)$$

where the abbreviation  $x^*(k)$  is used in place of  $x^*(k|k)^\dagger$ , and  $F(k)$  is the  $n \times n$  equivalent gain matrix which is determined so as to minimize the conditional expectation of the form

$$E\{\|\epsilon_1(x(k))\|^2 | Z_k\}, \quad (4.20)$$

where

$$\epsilon_1(x(k)) = f(x(k), k) - F(k)[x(k) - x^*(k)] - f(x^*(k), k). \quad (4.21)$$

By using Eq. (4.17), Eq. (4.19) is interpreted as

$$\int \|\epsilon_1(x(k))\|^2 p(x(k) | Z_k) dx(k). \quad (4.22)$$

It is easily derived from Eq. (4.20) that the equivalent gain matrix  $F(k)$  is given by

$$F(k) = E\{[f(x(k), k) - f(x^*(k), k)][x(k) - x^*(k)]' | Z_k\} M^{-1}(k). \quad (4.23)$$

From this equation, the equivalent gain is determined by the non-linearity  $f(x(k), k)$  and the a posteriori probability density function  $p(x(k) | Z_k)$ .

---

<sup>†</sup> For simplicity, if there is no confusions, this abbreviation will be used without notice in the following sections.



Now we shall consider the linearization of the observation system described by Eq. (4.2). In this case, as is seen from Eq. (4.16), the nonlinearity  $h(x(k+1), k+1)$  should be linearized on the basis of the information concerning the state  $x(k+1)$  conditioned on data  $Z_k$ . The information is contained in the a posteriori probability density function  $p(x(k+1)|Z_k)$ . By the Markov property, we have

$$p(x(k+1)|Z_k) = \int p(x(k+1)|x(k))p(x(k)|Z_k)dx(k). \quad (4.24)$$

Therefore, it follows from Eqs. (4.8) and (4.17) that

$$p(x(k+1)|Z_k) = C_5 \int \exp \left\{ -\frac{1}{2} \|x(k+1) - f(x(k), k)\|_Q^2 - \frac{1}{2} \|x(k) - x^*(k)\|_M^2 \right\} dx(k), \quad (4.25)$$

where  $C_5$  is a normalizing factor. Replacing the nonlinearity, in Eq. (4.25) by its approximation  $F(k)[x(k) - x^*(k)] + f(x^*(k), k)$  and performing the integration yields

$$p(x(k+1)|Z_k) = (2\pi)^{-n/2} |P(k+1)| \times \exp \left\{ -\frac{1}{2} \|x(k+1) - f(x^*(k), k)\|_P^2 \right\}, \quad (4.26)$$

where

$$P(k+1) = F(k)M(k)F'(k) + Q(k). \quad (4.27)$$

The derivation of Eqs. (4.26) and (4.27) is found in Appendix B at the end of this chapter. In Eq. (4.26), the  $f(x^*(k), k)$  can be regarded as the prediction of the state  $x(k+1)$  based on the data  $Z_k$ , i.e.,

$$x^*(k+1|k) = f(x^*(k), k). \quad (4.28)$$

Therefore, it follows that  $P(k+1)$  is the  $n \times n$  covariance matrix of the one step prediction error, that is,

$$P(k+1) = E\{[x(k+1) - x^*(k+1|k)][x(k+1) - x^*(k+1|k)]' | Z_k\}. \quad (4.29)$$

It is natural from Eq. (4.26) that Eq. (4.2), where  $k$  is replaced by  $k+1$ , is linearized as

$$\begin{aligned} z(k+1) \approx & H(k+1)[x(k+1) - x^*(k+1|k)] \\ & + h(x^*(k+1|k), k+1) + v(k+1), \end{aligned} \quad (4.30)$$

where  $H(k+1)$  is the  $p \times n$  equivalent gain matrix of the nonlinearity  $h(x(k+1), k+1)$  which is determined so as to minimize the conditional expectation

$$E\{\|\epsilon_2(x(k+1))\|^2 | Z_k\}, \quad (4.31)$$

where

$$\begin{aligned} \epsilon_2(x(k+1)) = & h(x(k+1), k+1) - H(k+1)[x(k+1) - x^*(k+1|k)] \\ & - h(x^*(k+1|k), k+1). \end{aligned} \quad (4.32)$$

The equivalent gain matrix is then given by

$$\begin{aligned} H(k+1) = & E\{[h(x(k+1), k+1) - h(x^*(k+1|k), k+1)] \\ & \times [x(k+1) - x^*(k+1|k)]' | Z_k\} P^{-1}(k+1). \end{aligned} \quad (4.33)$$

Hence, the approximations to the nonlinear systems are obtained. Now we shall proceed to the next step, that is, the derivation of the recursive scheme for the approximate estimator.

#### 4.5 Recursive Relations for Approximate Estimator

Replacing, in Eq. (4.9) for  $j = k+1$ , the nonlinearity  $h(x(k+1), k+1)$  by its approximation  $H(k+1)[x(k+1) - x^*(k+1|k)] + h(x^*(k+1|k), k+1)$ , we have

$$p(z(k+1)|x(k+1)) = C_6 \exp \left\{ -\frac{1}{2} \|z(k+1) - H(k+1)[x(k+1) - x^*(k+1|k)] - h(x^*(k+1|k), k+1)\|_{R^{-1}(k+1)}^2 \right\}, \quad (4.34)$$

where  $C_6$  is a normalizing factor. By using Eq. (4.15)<sub>1</sub>, Eq. (4.14) becomes

$$p(x(k+1)|Z_{k+1}) = \frac{p(z(k+1)|x(k+1))p(x(k+1)|Z_k)}{p(z(k+1)|Z_k)}. \quad (4.35)$$

Then, it follows from Eqs. (4.26), (4.34) and (4.35) that

$$p(x(k+1)|Z_{k+1}) = (2\pi)^{-n/2} |M(k+1)|^{-1/2} \times \exp \left\{ -\frac{1}{2} \|x(k+1) - x^*(k+1)\|_{M^{-1}(k+1)}^2 \right\}, \quad (4.36)$$

where

$$x^*(k+1) = x^*(k+1|k) + \Delta(k+1)[z(k+1) - h(x^*(k+1|k), k+1)], \quad (4.37)_1$$

$$= f(x^*(k), k) + \Delta(k+1)[z(k+1) - h(f(x^*(k), k), k+1)], \quad (4.37)_2$$

$$M(k+1) = [P^{-1}(k+1) + H'(k+1)R^{-1}(k+1)H(k+1)]^{-1}, \quad (4.38)_1$$

$$= P(k+1) - \Delta(k+1)H(k+1)P(k+1), \quad (4.38)_2$$

$$\Delta(k+1) = M(k+1)H'(k+1)R^{-1}(k+1), \quad (4.39)_1$$

$$= P(k+1)H'(k+1)[H(k+1)P(k+1)H'(k+1) + R(k+1)]^{-1}. \quad (4.39)_2$$

Eqs. (4.38) and (4.39) are usually referred to as the matrix inversion lemma (see Appendix A).

Applying the above result, the approximate solution of the estimation problem considered here is given by the following recursive scheme.

(i) The difference equation for the approximate estimator is

$$\mathbf{x}^*(k+1) = \mathbf{f}(\mathbf{x}^*(k), k) + \Delta(k+1)[\mathbf{z}(k+1) - \mathbf{h}(\mathbf{f}(\mathbf{x}^*(k), k), k+1)], \quad (4.40)$$

where

$$\Delta(k+1) = \mathbf{P}(k+1)\mathbf{H}'(k+1)[\mathbf{H}(k+1)\mathbf{P}(k+1)\mathbf{H}'(k+1) + \mathbf{R}(k+1)]^{-1}. \quad (4.41)$$

Note that  $\Delta(k+1)$  is the  $n \times p$  matrix which represents the gain of the updating term by the noisy observation.

(ii) The variance equations are, from Eqs. (4.27) and (4.38)<sub>2</sub>, given by

$$\mathbf{P}(k+1) = \mathbf{F}(k)\mathbf{M}(k)\mathbf{F}'(k) + \mathbf{Q}(k) \quad (4.42)$$

and

$$\mathbf{M}(k+1) = \mathbf{P}(k+1) - \Delta(k+1)\mathbf{H}(k+1)\mathbf{P}(k+1). \quad (4.43)$$

(iii) The equivalent gain matrices are

$$\mathbf{F}(k) = \mathbf{E}\{[\mathbf{f}(\mathbf{x}(k), k) - \mathbf{f}(\mathbf{x}^*(k), k)][\mathbf{x}(k) - \mathbf{x}^*(k)]' | \mathbf{Z}_k\} \mathbf{M}^{-1}(k) \quad (4.44)$$

and

$$\begin{aligned} \mathbf{H}(k+1) = \mathbf{E}\{[\mathbf{h}(\mathbf{x}(k+1), k+1) - \mathbf{h}(\mathbf{x}^*(k+1|k), k+1)] \\ \times [\mathbf{x}(k+1) - \mathbf{x}^*(k+1|k)]' | \mathbf{Z}_k\} \mathbf{P}^{-1}(k+1). \end{aligned} \quad (4.45)$$

The initial conditions to Eqs. (4.40) and (4.43) are given by  $\mathbf{x}^*(0) = 0$  and  $\mathbf{M}(0) = \mathbf{E}\{\mathbf{x}(0)\mathbf{x}'(0)\}$ .

The structure of the approximate nonlinear estimator is completely determined by Eqs. (4.41) to (4.45). Among them, Eqs. (4.42) to (4.45)

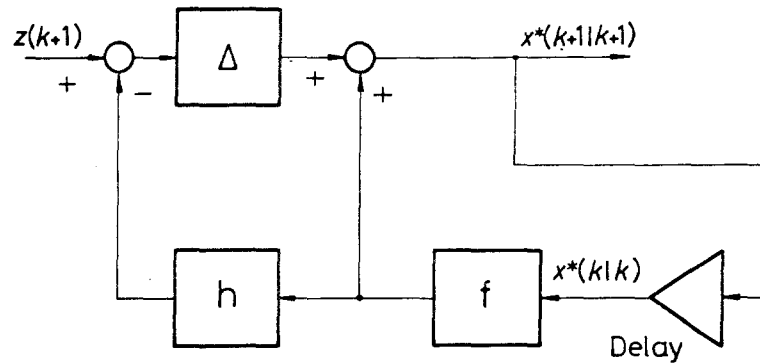


Fig. 4.2 Schematic representation of the nonlinear estimator

are necessary for the evaluation of the gain matrix  $\Delta(k+1)$ . The nonlinear estimator obtained is depicted in Fig. 4.2. It should be noted that if the system in question is linear, the structure of the nonlinear estimator coincides with that of the well-known Wiener-Kalman filter[26].

From Eqs. (4.44) and (4.45), both the equivalent gain matrices  $F(k)$  and  $H(k+1)$  become some functions of the observed data  $Z_k$ . Therefore, it is impossible to evaluate the covariance matrices  $M(k)$  and  $P(k+1)$  before using the filter in practice; this fact is the common feature to the nonlinear estimation problem. On the other hand, for the linear case, the error covariance matrices can be computed before the filter is actually realized.

It should be also noted that the resultant scheme does not contain the inverse matrices of  $Q(k)$  and  $R(k+1)$ . This fact implies that the assumption that both the noise covariance matrices are nonsingular is not necessary for the application of the resultant scheme to the actual problem. Hence, the assumption that  $Q(k)$  and  $R(k+1)$  are both nonsingular

is removed.

In fact, this is proved as follows. Consider the case where both the covariance matrices  $Q(k)$  and  $R(k)$  are singular. In this case,  $Q^{-1}(k)$  and  $R^{-1}(k)$  are of no meaning, so that the expressions Eqs. (4.8) and (4.9) is impossible. Let

$$\tilde{Q}(k) = Q(k) + \lambda_1 I_n \quad (4.46)$$

and

$$\tilde{R}(k) = R(k) + \lambda_2 I_p, \quad (4.47)$$

where  $I_n$  and  $I_p$  are  $n \times n$  and  $p \times p$  unit matrices, respectively, and  $\lambda_1$ ,  $\lambda_2$  are positive numbers. It is clear that  $\tilde{Q}(k)$  and  $\tilde{R}(k)$  are both positive definite; therefore, the respective inverses exist. If we replace  $Q(k)$  and  $R(k)$ , in Eqs. (4.8) and (4.9), by  $\tilde{Q}(k)$  and  $\tilde{R}(k)$ , respectively, we have

$$\Delta(k+1) = P(k+1)H'(k+1)[H(k+1)P(k+1)H'(k+1) + \tilde{R}(k+1)]^{-1} \quad (4.48)$$

and

$$P(k+1) = F(k)M(k)F'(k) + \tilde{Q}(k), \quad (4.49)$$

in place of Eqs. (4.41) and (4.42). Taking the limits  $\lambda_1 \rightarrow 0$  and  $\lambda_2 \rightarrow 0$ , we have  $\tilde{Q}(k) \rightarrow Q(k)$  and  $\tilde{R}(k) \rightarrow R(k)$ . Hence, Eqs. (4.48) and (4.49) approach to Eqs. (4.41) and (4.42), respectively, as was to be proved.

#### 4.6 Concluding Remarks

In this chapter, the problem of estimating the state variables for noisy nonlinear dynamical systems has been considered. The approximation technique which is applicable to the nonlinear noisy observation problem

is obtained by using the Bayes theorem and the concept of statistical linearization technique. The key assumption involved is that the a posteriori probability density function  $p(x(k)|Z_k)$  of the state conditioned on the observed data  $Z_k$  is Gaussian.

The accuracy consideration of the approximation method developed in this chapter will be presented in Chapter 5, including digital simulation studies.

#### Appendix A: Matrix Inversion Lemma (See for example [Aoki, 1])

Let  $P$  be an  $n \times n$  matrix,  $R$  be a  $p \times p$  matrix, and  $H$  be a  $p \times n$  matrix. If  $P$  and  $R$  are both symmetric and nonnegative definite, then the matrix identities hold:

$$[P^{-1} + H'R^{-1}H]^{-1} = P - PH'[HPH' + R]^{-1}HP \quad (A.1)$$

and

$$[P^{-1} + H'R^{-1}H]^{-1}H'R^{-1} = PH'[HPH' + R]^{-1}. \quad (A.2)$$

Proof: To prove Eq. (A.1), it suffices to show

$$[P^{-1} + H'R^{-1}H][P - PH'[HPH' + R]^{-1}HP] = I \text{ (unit matrix)}. \quad (A.3)$$

In fact,

$$\begin{aligned} \text{L.H.S. of Eq. (A.3)} &= \\ &= I + H'R^{-1}HP - H'[HPH' + R]^{-1}HP - H'R^{-1}HPH'[HPH' + R]^{-1}HP \\ &= I + H'R^{-1}\{I - R[HPH' + R]^{-1} - HPH'[HPH' + R]^{-1}\}HP \\ &= I + H'R^{-1}\{I - [HPH' + R][HPH' + R]^{-1}\}HP \\ &= I. \end{aligned}$$

By using Eq. (A. 1), Eq. (A.2) can be proved as follows.

$$\begin{aligned}
& [P^{-1} + H'R^{-1}H]^{-1}H'R^{-1} = \\
& = \{P - PH'[HPH' + R]^{-1}HP\}H'R^{-1} \\
& = PH'\{I - [HPH' + R]^{-1}HPH'\}R^{-1} \\
& = PH'\{[HPH' + R]^{-1}[HPH' + R] - [HPH' + R]^{-1}HPH'\}R^{-1} \\
& = PH'[HPH' + R]^{-1}[HPH' + R - HPH']R^{-1} \\
& = PH'[HPH' + R]^{-1}.
\end{aligned}$$

The proof is finished.

#### Appendix B: Derivation of Eqs. (4.26) and (4.27)

Let the curly bracket in Eq. (4.25) be  $L$ . By using the approximation to the nonlinearity  $f(x(k), k)$ , we have

$$\begin{aligned}
-2L &= \|x(k+1) - F(k)[x(k) - x^*(k)] - f(x^*(k), k)\|_Q^{-1}(k)^2 \\
&\quad + \|x(k) - x^*(k)\|_M^{-1}(k)^2.
\end{aligned} \tag{B.1}$$

Letting

$$\bar{x} = x(k) - x^*(k) \tag{B.2}$$

and

$$\bar{y} = x(k+1) - f(x^*(k), k), \tag{B.3}$$

Eq. (B.1) becomes

$$-2L = \|\bar{y} - F\bar{x}\|_Q^{-1}^2 + \|\bar{y}\|_M^{-1}^2. \tag{B.4}$$

Rearranging Eq. (B.4), it follows that



$$-2L = \|\bar{x} - \bar{z}\|_V^{-1} + \|\bar{w}\|_P^{-1}. \quad (B.5)$$

where

$$V^{-1} = F'Q^{-1}F + M, \quad (B.6)$$

$$\bar{z} = VF'Q^{-1}\bar{y}, \quad (B.7)$$

$$P^{-1} = Q^{-1} - Q^{-1}FVF'Q^{-1}, \quad (B.8)$$

$$\bar{w} = \bar{y}. \quad (B.9)$$

From Eq. (B.2), the integration of Eq. (4.25) with respect to  $x(k)$  is equal to the integration with respect to  $\bar{x}$ . Therefore, by using Eq. (B.5), we have

$$p(x(k+1)|Z_k) = (2\pi)^{-n/2} |P|^{-1/2} \exp\left\{-\frac{1}{2}\|\bar{y}\|_P^{-1}\right\}. \quad (B.10)$$

Furthermore, from Eq. (B.8),  $P$  becomes

$$P = [Q^{-1} - Q^{-1}FVF'Q^{-1}]^{-1}, \quad (B.11)$$

so that, by using the matrix identity Eq. (A.1), we finally obtain

$$P = FMF' + Q, \quad (B.12)$$

as was to be proved.

## CHAPTER 5

### ACCURACY CONSIDERATION FOR APPROXIMATE STATE ESTIMATION TECHNIQUE

#### 5.1 Introduction and Problem Statement

In Chapter 4, we have presented an approximation technique of estimating state variables for noisy discrete-time nonlinear dynamical systems. The principal approach of the technique is the application of the concept of statistical linearization for nonlinear systems under the assumption that the a posteriori probability density function of the state is Gaussian. Therefore, the accuracy of the approximate estimation technique mainly depends upon whether or not the a posteriori probability density function can adequately be approximated by a Gaussian density function. Hence, it is an important problem to examine the assumption that the a posteriori probability density function is nearly Gaussian.

In this chapter, for a typical first order system, the correctness of the assumption is tested by means of two numerical examples. The numerical versions of the a posteriori probability density function are evaluated by using a digital computer.

We consider a first order system whose dynamical characteristic is expressed by the first order nonlinear difference equation

$$x(k+1) = f(x(k), k) + w(k), \quad (5.1)$$

where  $x(k)$  is a state variable, and  $w(k)$  is a Gaussian white noise with  $N(0, q)$ . The output observation is the nonlinear combination of the state variable with additive noise, i.e.,

$$z(k) = h(x(k), k) + v(k), \quad (5.2)$$

where  $z(k)$  is an observed signal, and  $v(k)$  is a Gaussian white noise with  $N(0, r)$  and is independent of the  $w(k)$ . It is assumed that  $f(x(k), k)$  and  $h(x(k), k)$  are nonlinear functions of the forms

$$f(x(k), k) = \alpha x(k) + \beta x^2(k) \quad (5.3)$$

and

$$h(x(k), k) = x(k) + \lambda x^3(k), \quad (5.4)$$

where  $\alpha$ ,  $\beta$  and  $\lambda$  are all known constants.

## 5.2 Approximate State Estimation Scheme

By applying the approximate estimation technique developed in Chapter 4, we have the following scheme.

- (i) The difference equation for the estimator is

$$x^*(k+1) = f(x^*(k)) + \Delta(k+1)[z(k+1) - h(f(x^*(k)))], \quad (5.5)$$

where the abbreviations  $f(x^*(k))$  and  $h(f(x^*(k)))$  are used in stead of  $f(x^*(k), k)$  and  $h(f(x^*(k), k), k+1)$ , respectively.

(ii) The gain of the updating term is

$$\Delta(k+1) = P(k+1)H(k+1)/[r + H^2(k+1)P(k+1)]. \quad (5.6)$$

(iii) The variances of estimation errors are given by

$$P(k+1) = F^2(k)M(k) + q \quad (5.7)$$

and

$$M(k+1) = P(k+1) - \Delta(k+1)H(k+1)P(k+1), \quad (5.8)$$

where  $M(k)$  is the variance of the filtered estimation error, and  $P(k+1)$  is the variance of one step predicted estimation error.

(iv) The equivalent gain of the nonlinearities  $f$  and  $h$  are given by

$$F(k) = \alpha + 2\beta x^*(k) \quad (5.9)$$

and

$$H(k+1) = 1.0 + 3\lambda\{P(k+1) + [f(x^*(k))]^2\}, \quad (5.10)$$

respectively.

The initial conditions to Eqs. (5.5) and (5.7) are  $x^*(0) = 0$  and  $M(0) = E\{x^2(0)\}$ , respectively.

Typical running values of the estimate  $x^*(k)$ , together with the state  $x(k)$  and the observed signal  $z(k)$ , are shown in Fig. 5.1. The following numerical values are used in the computation:

$$\alpha = 0.9, \quad \beta = 0.05, \quad \lambda = 0.0, \quad q = 0.06, \quad r = 0.04,$$

$$x(0) = 0.2, \quad x^*(0) = 0, \quad M(0) = 1.0.$$

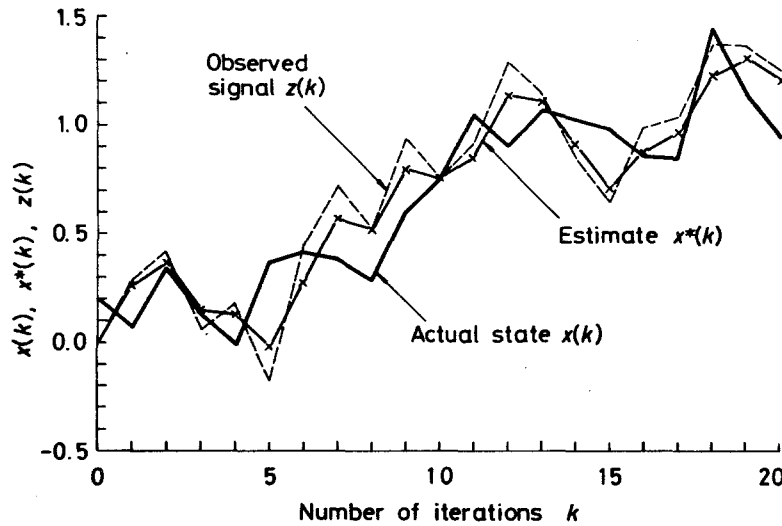


Fig. 5.1 Sample records of the processes  $x(k)$ ,  $z(k)$ , and  $x^*(k)$

### 5.3 Exact Expression of the a Posteriori Probability Density Function

By making use of Eq. (4.16), we have a recurrence relation for the a posteriori probability density function of the state variable conditioned on the observed data as follows.

$$p(x(k+1)|Z_{k+1}) = C_1 \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2r}[z(k+1) - h(x(k+1))]^2\right. \\ \left.- \frac{1}{2q}[x(k+1) - f(x(k))]^2\right\} p(x(k)|Z_k) dx(k), \quad (5.11)$$

where  $C_1$  is a normalizing factor. As is already noted in Chapter 4, the integration of the right hand side of Eq. (5.11) is very difficult due to the existence of the nonlinearities. It is possible, however, to evaluate it numerically by a high speed digital computer; the result

will give a more accurate estimate of the state variable and also the shape of the a posteriori probability density function to be compared with the approximate Gaussian density with mean  $x^*(k)$  and variance  $M(k)$ .

For the numerical computation, it is assumed that the  $p(x(k)|Z_k)$  is distributed over  $[a, b]$ , neglecting the contribution in the outside of the interval. Furthermore, the interval  $[a, b]$  is partitioned into  $N$  subintervals with length  $\Delta$ , as shown in Fig. 5.2. Therefore, the  $p(x(k)|Z_k)$  can be approximated by

$$p(x(k+1)|Z_{k+1}) \approx C_2 \exp\left\{-\frac{1}{2r}[z(k+1) - h(x(k+1))]^2\right\} \times \sum_{i=0}^N \exp\left\{-\frac{1}{2q}[x(k+1) - f(\xi_i)]^2\right\} p(\xi_i|Z_k), \quad (5.12)$$

where

$$\xi_i = a + i(b - a)/N, \quad i = 0, 1, \dots, N.$$

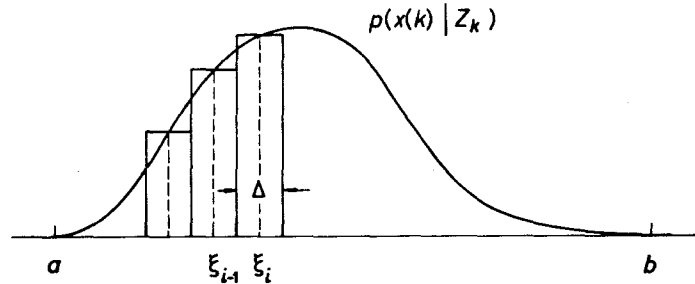


Fig. 5.2 Staircase approximation of a posteriori probability density function

#### 5.4 Digital Simulation Studies

##### 5.4.1 Nonlinear Dynamical System with Linear Observation

First, consider the case where the dynamical system is nonlinear and the observation system is linear. The following numerical values are used in the computation:

$$\alpha = 0.9, \quad \beta = 0.05, \quad \lambda = 0.0, \quad q = 0.06, \quad r = 0.04,$$

$$x(0) = 0.2, \quad x^*(0) = 0.0,$$

$$p(x(0)|Z_0) \sim N(0, 1).$$

Figs. 5.3 and 5.4 display the behavior of the absolute probability density function of the state of the nonlinear dynamical system described by Eq. (5.1). The skewness  $\gamma_1$  and kurtosis  $\gamma_2$  are coefficients of the orthogonal expansion of any 1-dimensional probability density function, that is, any probability density function with mean zero and unit variance can be expanded in the form[10]:

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} \left\{1 - \frac{\gamma_1}{3!}[3x - x^3] + \frac{\gamma_2}{4!}[3 - 6x^2 + x^4] + \dots\right\}, \quad (5.13)$$

so that the probability density function  $p(\zeta)$  with mean  $\mu$  and variance  $\sigma^2$  can be expressed by the substitution  $x = (\zeta - \mu)/\sigma$ . Let  $\mu_i$  be the  $i$ -th moment of  $p(\zeta)$ , then it follows that

$$\gamma_1 = \mu_3/\sigma^3 \quad (5.14)_1$$

and

$$\gamma_2 = \mu_4/\sigma^4 - 3. \quad (5.14)_2$$

If both the skewness and kurtosis are sufficiently small, the probability density function is well approximated by a Gaussian density function.

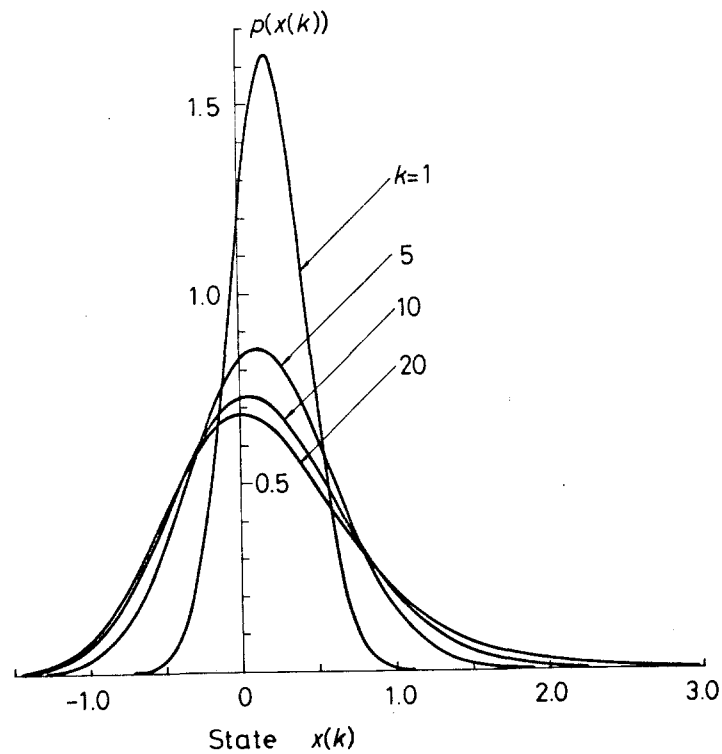


Fig. 5.3 Transition of the absolute probability density functions of the state  $x(k)$  of the nonlinear system described by Eq. (5.1)

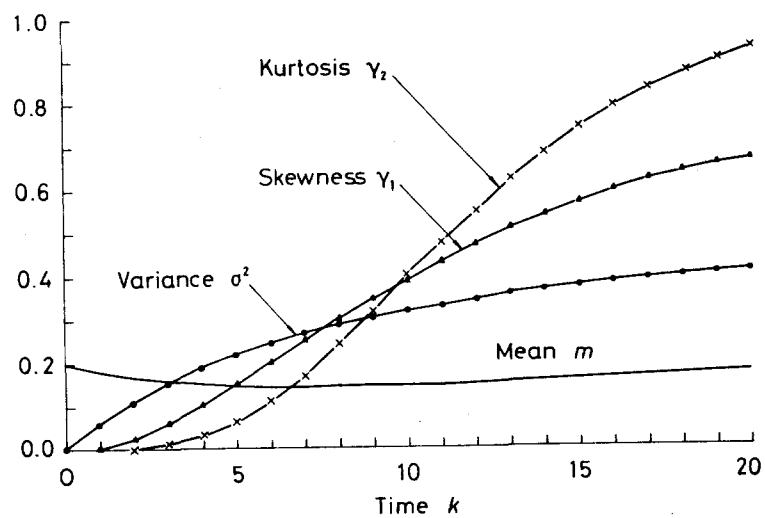


Fig. 5.4 Variation of various moments of the a posteriori probability density function shown in Fig. 5.3



Table 5.1 shows the comparison of the results obtained by the approximate method in section 5.2 and the numerical method developed in section 5.3, where  $\hat{x}(k)$  and  $\sigma^2(k)$  represent the a posteriori conditional mean and variance, respectively. It is found from this table that there is no distinguished differences in the two procedure.

k	$\hat{x}(k)$	$x^*(k)$	$\sigma^2(k)$	M(k)
0	0.000000	0.000000	1.000000	1.000000
1	0.273291	0.276284	0.038379	0.038242
2	0.368785	0.369371	0.028016	0.027964
3	0.150237	0.150080	0.027108	0.027156
4	0.145776	0.145542	0.026962	0.026967
5	-0.070909	-0.071381	0.026895	0.026947
6	0.284610	0.283828	0.026908	0.026832
7	0.574803	0.573886	0.027090	0.027008
8	0.524121	0.523673	0.027180	0.027177
9	0.803411	0.802664	0.027245	0.027167
10	0.762359	0.761970	0.027321	0.027315
11	0.853912	0.853555	0.027341	0.027307
12	1.144457	1.143645	0.027445	0.027356
13	1.124994	1.124591	0.027531	0.027517
14	0.924023	0.923640	0.027481	0.027522
15	0.722079	0.721663	0.027368	0.027414
16	0.889529	0.889083	0.027343	0.027295
17	0.977886	0.977501	0.027411	0.027373
18	1.244916	1.244166	0.027511	0.027429
19	1.321993	1.321548	0.027604	0.027579
20	1.266196	1.265933	0.027630	0.027636

Table 5.1 Comparison of the approximate and a posteriori estimates for nonlinear dynamical system with linear observation

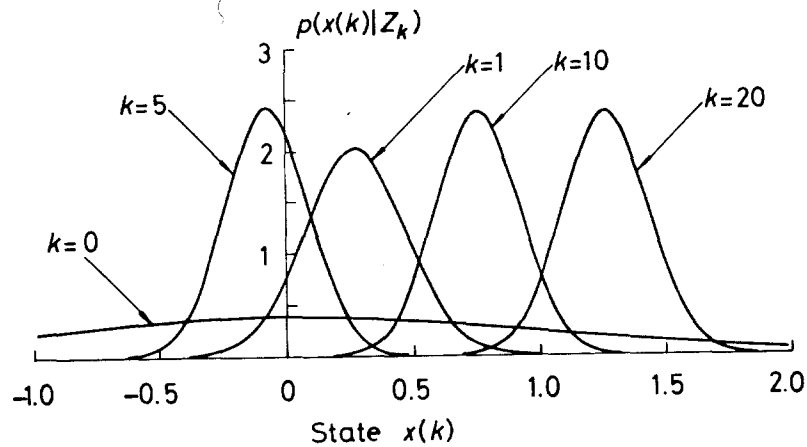


Fig. 5.5 Transition of the a posteriori probability density functions

Fig. 5.5 displays the transition of the a posteriori probability density functions  $p(x(k)|Z_k)$  which are evaluated through Eq. (5.12), where  $a = -1.0$ ,  $b = 2.0$ ,  $N = 150$ , and  $\Delta = 0.02$ . In this case, the skewness  $\gamma_1$  and kurtosis  $\gamma_2$  are of order  $10^{-3}$ ; that is, the distribution is almost Gaussian.

#### 5.4.2 Linear System with Nonlinear Observation

Let us now consider the case where the dynamical system is linear, whereas the observation system is nonlinear. The numerical values used in the computation are the same as the previous example except for  $\beta = 0.0$  and  $\lambda = -0.1$ .

Table 5.2 shows the comparison of the results obtained by the approximate state estimation technique and by the numerical method developed in section 5.3.

k	$\hat{x}(k)$	$x^*(k)$	$\sigma^2(k)$	M(k)
0	0.000000	0.000000	1.000000	1.000000
1	0.286940	0.358229	0.042461	0.067557
2	0.375768	0.394884	0.030523	0.032741
3	0.148043	0.146950	0.028154	0.029883
4	0.141834	0.139405	0.027818	0.028272
5	-0.081419	-0.083764	0.027506	0.028089
6	0.270808	0.271571	0.027844	0.027955
7	0.561277	0.556597	0.030375	0.028566
8	0.514093	0.503036	0.031296	0.030994
9	0.789502	0.776390	0.034600	0.030689
10	0.734846	0.714016	0.036225	0.034574
11	0.766570	0.743522	0.036722	0.034002
12	1.048351	1.027334	0.041194	0.034441
13	1.011307	0.973853	0.046073	0.041268
14	0.754134	0.723176	0.039723	0.040905
15	0.496550	0.473878	0.032448	0.034985
16	0.652066	0.636837	0.033013	0.030838
17	0.737042	0.717528	0.035124	0.032314
18	0.909871	0.885531	0.038984	0.033758
19	0.999200	0.965667	0.043289	0.037338
20	0.952086	0.912797	0.044653	0.039995

Table 5.2 Comparison of the approximate and a posteriori estimates for linear dynamical system with nonlinear observation

Fig. 5.6 displays the a posteriori conditional probability density function evaluated by using Eq. (5.12), and Fig. 5.7 displays the variation of the skewness  $\gamma_1$  and kurtosis  $\gamma_2$  of the a posteriori probability density function shown in Fig. 5.6. The skewness and kurtosis are rather large compared with the previous example.

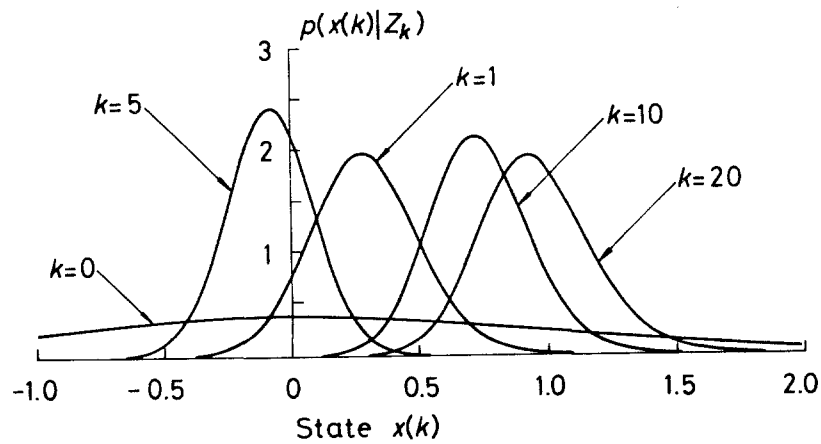


Fig. 5.6 Transition of the a posteriori probability density function

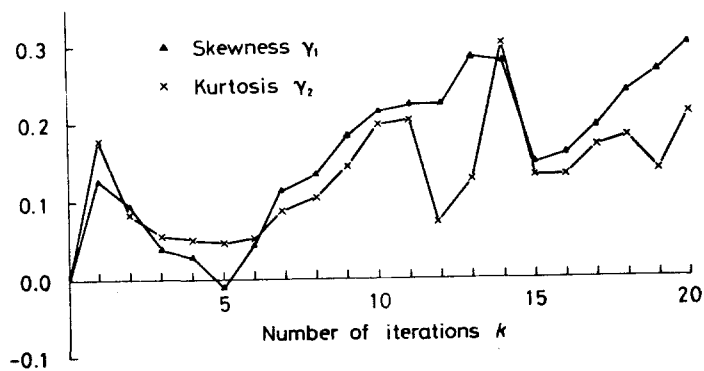


Fig. 5.7 Skewness and kurtosis of the a posteriori probability density function shown in Fig. 5.6

### 5.5 Concluding Remarks

It is impossible to make irrefutable statements on the accuracy of the approximate nonlinear state estimation technique on the basis of these two examples. Several conclusions can, however, be drawn from these results.

The assumption that the a posteriori probability density function is nearly Gaussian is a reasonable for the first order nonlinear systems with an additive Gaussian white noise and an additive observation noise. That is, in the present digital simulation studies, even if the controlled system is nonlinear, and if the observation system is linear, then the a posteriori probability density function of interest is almost Gaussian. Conversely, if the controlled system is linear and if the observation system is nonlinear, then the a posteriori probability density function slightly deviates from Gaussian. However, the numerical results obtained by the approximate method and by the numerical method developed in section 5.3 are nearly equal as shown in Table 5.2.

Digital simulation studies similar to this chapter are also found in Chapter 7, where 2-dimensional a posteriori probability densities are evaluated by a digital computer.

## CHAPTER 6

### ESTIMATION OF UNKNOWN PARAMETERS FOR NONLINEAR DYNAMICAL SYSTEMS

#### 6.1 Introduction

As is well-known, the identification or the parameter estimation problem is one of the most important aspects in the field of automatic control. A determination of a mathematical model of the system to be controlled, from its input and output data, is required to realize an optimal control or an adaptive control. With the development of high speed digital computers, it has become possible to construct a more accurate model of the system to be controlled on the basis of the data which is obtainable under the normal operation.

Until the present, a number of techniques for the identification or the parameter estimation of linear dynamical systems have been reported;

for example, Kalman[22] and Levin[34] treated the problem from the view point of least square; Kushner[30] presented a computationaly feasible method; Ho & Lee[19] applied the stochastic approximation method to the identification problem. A review of the identification problem has recently been reported by Eykhoff et al[14].

On the other hand, there have also been presented several papers on the nonlinear identification problems, which were first developed by Wiener[54], and later by Balakrishnan[3], etc. Cox[9] and Detchmندی & Sridhar[11] treated the state and parameter estimation problem by using the method of dynamic programming and the invariant embedding, respectively; Fukao[16] and Kumar[29] also considered the problem by using the Bayesian learning and the quasi-linearization, respectively.

In this chapter, we shall be concerned with the problem of estimating unknown constant parameters for nonlinear dynamical systems. The class of systems considered are those in which the dynamical characteristic is described by a vector nonlinear difference equation with unknown constant parameters; and the output observation is a nonlinear combination of state variables corrupted by additive noise.

In general, if the a posteriori probability density function of the unknown parameters is computed whenever the available data is obtained, then the solution of the estimation problem follows immediately, as was shown in Chapter 1. However, this is not the case for the class of problems considered here, because it is shown from the Bayesian point of view that the parameter estimation and the state estimation can not be separated. Therefore, the evaluation of the estimates of the unknown

parameters is quite difficult due to the fact that the form of the joint a posteriori probability density function of the unknown parameters and the state variables is highly complex.

The main purpose of this chapter is to give an approximation technique for the nonlinear parameter estimation problem by extending the method of state estimation developed in Chapter 4[41].

## 6.2 Statement of Problem

We shall consider the problem of estimating unknown parameters for noisy discrete-time nonlinear dynamical systems. The general formulation of the problem is given in this section. Consider a process described by the vector difference equation of the form

$$x(k+1) = f(x(k), \theta, k) + w(k), \quad (6.1)$$

where

$x(k)$  is an  $n$ -dimensional state vector,

$\theta$  is an  $m$ -dimensional vector denoting the unknown parameters,

$w(k)$  is an  $n$ -dimensional vector denoting the random input,

$f$  is an  $n$ -dimensional vector-valued nonlinear function.

Let the output observation be nonlinear combination of state variables corrupted by additive noise, i.e.,

$$z(k) = h(x(k), k) + v(k), \quad (6.2)$$

where

$z(k)$  is a  $p$ -dimensional vector denoting the observation( $p \leq n$ ),

$v(k)$  is a  $p$ -dimensional vector denoting the observation noise,



$h$  is a  $p$ -dimensional vector-valued nonlinear function.

It is assumed that the initial state  $x(0)$  is Gaussian and is independent of the  $w(k)$  and  $v(k)$ , and that the  $w(k)$  and  $v(k)$  are mutually independent Gaussian white noises with means zero and covariance matrices  $Q(k)$  and  $R(k)$ , respectively. It is also assumed for simplicity that the  $n \times n$  and  $p \times p$  covariance matrices  $Q(k)$  and  $R(k)$  are both positive definite. However, it will become clear from the discussion below that this assumption is not necessary.

The parameter estimation problem considered here is stated as follows: on the basis of the observed data  $Z_k = \{z(1), \dots, z(k)\}$ , find the best estimate  $\hat{\theta}$  of the unknown parameter  $\theta$  which minimizes the conditional expectation of the quadratic loss function

$$E\{\|\theta - \hat{\theta}\|^2 | Z_k\}. \quad (6.3)$$

The best estimate  $\hat{\theta}$  generally depends upon time  $k$ , so that we shall write  $\hat{\theta}(k)$  instead. Since, as mentioned in Chapter 1, the best estimate is given by the conditional expectation

$$\hat{\theta}(k) = E\{\theta | Z_k\}, \quad (6.4)$$

the evaluation of the a posteriori probability density function  $p(\theta | Z_k)$  of the unknown parameter  $\theta$  is also a part and parcel in the parameter estimation as in the state estimation considered in Chapter 4. By using the a posteriori probability density function  $p(\theta | Z_k)$ , Eq. (6.4) can be expressed as

$$\hat{\theta}(k) = \int \theta p(\theta | Z_k) d\theta. \quad (6.5)$$

### 6.3 A Posteriori Probability Density Function and General Discussion

By making use of the Bayes theorem[18], the a posteriori probability density function of the unknown parameter vector  $\theta$  conditioned on the observed data  $Z_k$  can be expressed as

$$p(\theta|Z_k) = \frac{p(Z_k|\theta)p(\theta)}{p(Z_k)}. \quad (6.6)$$

The term of significance is the numerator of the right hand side of this equation; the  $p(\theta)$  is the a priori probability density function of the unknown parameter vector  $\theta$ ; the  $p(Z_k|\theta)$  is the conditional probability density function of  $Z_k$  given  $\theta$ , and is expressed as

$$p(Z_k|\theta) = \int p(Z_k|X_k, \theta)p(X_k|\theta)d\Omega(X_k), \quad (6.7)$$

where  $X_k = \{x(0), \dots, x(k)\}$  and the expression  $d\Omega(X_k)$  is the elementary volume of the arguments. In Eq. (6.7), the  $p(Z_k|X_k, \theta)$  is the conditional probability density function of  $Z_k$  given  $X_k$  and  $\theta$ ; the  $p(X_k|\theta)$  is the conditional probability density function of  $X_k$  given  $\theta$ .

Since, in Eqs. (6.1) and (6.2), both the  $w(k)$  and  $v(k)$  are Gaussian white, the  $x(k)$  and  $z(k)$  become Markov processes. Therefore, it follows that

$$p(Z_k|X_k, \theta) = \prod_{j=1}^k p(z(j)|x(j), \theta) \quad (6.8)$$

and

$$p(X_k|\theta) = p(x(0)) \prod_{j=0}^{k-1} p(x(j+1)|x(j), \theta), \quad (6.9)$$

where  $p(x(0))$  is the a priori probability density function of the initial state  $x(0)$ . It is easily verified that the  $p(z(j)|x(j), \theta)$  is Gaussian with mean  $h(x(j), j)$  and covariance matrix  $R(j)$  and that the  $p(x(j+1)|x(j), \theta)$  is also Gaussian with mean  $f(x(j), \theta, j)$  and covariance matrix  $Q(j)$ , i.e.,

$$p(z(j)|x(j), \theta) = C_1 \exp \left\{ -\frac{1}{2} \|z(j) - h(x(j), j)\|_{R^{-1}(j)}^2 \right\} \quad (6.10)$$

and

$$p(x(j+1)|x(j), \theta) = C_2 \exp \left\{ -\frac{1}{2} \|x(j+1) - f(x(j), \theta, j)\|_{Q^{-1}(j)}^2 \right\} \quad (6.11)$$

where  $C_1$  and  $C_2$  are normalizing factors†. Substituting Eqs. (6.10) and (6.11) into Eqs. (6.8) and (6.9), respectively, we have

$$p(Z_k|X_k, \theta) = C_3 \exp \left\{ -\frac{1}{2} \sum_{j=1}^k \|z(j) - h(x(j), j)\|_{R^{-1}(j)}^2 \right\} \quad (6.12)$$

and

$$p(X_k|\theta) = C_4 \exp \left\{ -\frac{1}{2} \sum_{j=0}^{k-1} \|x(j+1) - f(x(j), \theta, j)\|_{Q^{-1}(j)}^2 - \frac{1}{2} \|x(0)\|_{M^{-1}(0)}^2 \right\} \quad (6.13)$$

where a Gaussian distribution with mean zero and covariance matrix  $M(0)$  is assigned to the a priori distribution of the initial state  $x(0)$ ; the  $M(0)$  is assumed to be  $n \times n$  positive definite matrix.

Applying Eqs. (6.7), (6.12) and (6.13) to Eq. (6.6) yields

$$p(\theta|Z_k) = C_5 \int \exp \left\{ -\frac{1}{2} \sum_{j=1}^k \|z(j) - h(x(j), j)\|_{R^{-1}(j)}^2 - \right.$$

---

†  $C_1, C_2, \dots$ , are normalizing factors throughout.

$$\begin{aligned}
& - \frac{1}{2} \sum_{j=0}^{k-1} \|x(j+1) - f(x(j), \theta, j)\|_{Q^{-1}(j)}^2 \\
& - \frac{1}{2} \|x(0)\|_{M^{-1}(0)}^2 - \frac{1}{2} \|\theta\|_{V^{-1}(0)}^2 \} d\Omega(x_k), \quad (6.14)
\end{aligned}$$

where the a priori probability density function  $p(\theta)$  is assumed to be Gaussian with mean zero and covariance matrix  $V(0)$  which is an  $m \times m$  positive definite matrix. Eq. (6.14) is the a posteriori probability density function of the unknown parameter vector  $\theta$  conditioned on the observed data  $Z_k$ . Applying Eq. (6.14) to Eq. (6.5), we have the best estimate  $\hat{\theta}(k)$  in the sense of least mean square. However, the following example shows how the evaluation of the right hand side of Eq. (6.5) is difficult even though the system is the simplest one.

Example Consider a system described by the scalar equations

$$x(k+1) = \alpha x(k) + w(k) \quad (6.15)$$

and

$$z(k) = x(k) + v(k), \quad (6.16)$$

where  $\alpha$  is an unknown parameter to be estimated; the  $w(k)$  and  $v(k)$  are mutually independent Gaussian white noises with mean zero and variances  $q$  and  $r$ , respectively; and the a priori probability density functions of  $\alpha$  and  $x(0)$  are assumed to be

$$p(\alpha) = (2\pi\sigma_\alpha^2)^{-1/2} \exp\left\{-\frac{1}{2\sigma_\alpha^2}(\alpha - \bar{\alpha})^2\right\} \quad (6.17)$$

and

$$p(x(0)) = (2\pi\sigma_x^2)^{-1/2} \exp\left\{-\frac{1}{2\sigma_x^2}(x(0) - \overline{x(0)})^2\right\}. \quad (6.18)$$

Let us consider the problem of estimating the parameter  $\alpha$  on the basis of the observation  $z(1)$ , that is, the single stage estimation problem. By the procedure similar to the derivation of Eq. (6.14), we have

$$p(\alpha|Z_1) = C_6 \{\alpha^2 \sigma_x^2 + q + r\}^{1/2} \exp \left\{ -\frac{1}{2\sigma_\alpha^2} [\alpha - \bar{\alpha}]^2 - \frac{z^2(1)}{2[\alpha^2 \sigma_x^2 + q + r]} + \frac{\alpha z(1) \overline{x(0)}}{\alpha^2 \sigma_x^2 + q + r} + \frac{r \alpha^2 \overline{x(0)}^2}{[\alpha^2 \sigma_x^2 + q][\alpha^2 \sigma_x^2 + q + r]} \right\}. \quad (6.19)$$

Therefore, from Eq. (6.5), it follows that

$$\hat{\alpha}(1) = \int_{-\infty}^{\infty} \alpha p(\alpha|Z_1) d\alpha. \quad (6.20)$$

It is then clear that as the number of observations  $k$  increases, the evaluation of  $p(\alpha|Z_k)$  and therefore the evaluation  $\hat{\alpha}(k)$  becomes more and more difficult. Furthermore, the above nonsequential procedure is inconvenient because a repetition of the entire calculation is required whenever the available data is obtained.

Now consider the evolution of the a posteriori probability density function with the increase of the observations, namely, the relation between  $p(\theta|Z_k)$  and  $p(\theta|Z_{k+1})$ . It follows from the Bayes theorem that

$$p(\theta|Z_{k+1}) = \frac{\int p(z(k+1), x(k+1), \theta|Z_k) dx(k+1)}{p(z(k+1)|Z_k)} \quad (6.21)$$

and

$$\begin{aligned}
& p(z(k+1), x(k+1), \theta | Z_k) \\
& = \int p(z(k+1) | x(k+1), \theta) p(x(k+1) | x(k), \theta) p(x(k), \theta | Z_k) dx(k). \quad (6.22)
\end{aligned}$$

Application of Eqs. (6.10), (6.11) and (6.22) to Eq. (6.20) gives

$$\begin{aligned}
p(\theta | Z_{k+1}) = C_7 \iint \exp \left\{ -\frac{1}{2} \| z(k+1) - h(x(k+1), k+1) \|_{R^{-1}(k+1)}^2 \right. \\
\left. - \frac{1}{2} \| x(k+1) - f(x(k), \theta, k) \|_{Q^{-1}(k)}^2 \right\} p(x(k), \theta | Z_k) dx(k) dx(k+1). \quad (6.23)
\end{aligned}$$

This equation is not a recursive relation for the a posteriori probability density function  $p(\theta | Z_k)$ , because the joint a posteriori probability density function  $p(x(k), \theta | Z_k)$  is contained in the right hand side. Hence, Eq. (6.23) is not suitable for the derivation of the recursive relation. However, we can derive the recursive relation for the joint a posteriori probability density function  $p(x(k), \theta | Z_k)$ . In fact, since the left hand side of Eq. (6.23) is expressed as

$$p(\theta | Z_{k+1}) = \int p(x(k+1), \theta | Z_{k+1}) dx(k+1), \quad (6.24)$$

it follows that

$$\begin{aligned}
p(x(k+1), \theta | Z_{k+1}) = C_7 \int \exp \left\{ -\frac{1}{2} \| z(k+1) - h(x(k+1), k+1) \|_{R^{-1}(k+1)}^2 \right. \\
\left. - \frac{1}{2} \| x(k+1) - f(x(k), \theta, k) \|_{Q^{-1}(k)}^2 \right\} p(x(k), \theta | Z_k) dx(k). \quad (6.25)
\end{aligned}$$

This equation can be interpreted as the difference-integral equation

showing how the joint a posteriori probability density function of the state and the unknown parameter evolves with the increase of observations. Therefore, to obtain a recursive estimator, the evaluation of the joint a posteriori probability density function  $p(x(k), \theta | Z_k)$  of the state and the unknown parameter is inevitable for the class of problems considered here. In other words, the parameter estimation problem can not be treated separately from the state estimation problem. From this point of view, we can conclude that it is convenient and desirable to embed the parameter estimation problem in the state estimation problem with the enlarged state vector, which will be defined later; because the information about the unknown parameter  $\theta$  contained in  $p(\theta | Z_k)$  is not more than that contained in  $p(x(k), \theta | Z_k)$ . Therefore, Eq. (6.25) comes to play an important role in the parameter estimation problem considered here.

Since the unknown parameter  $\theta$  is constant, it is possible to assume that  $\theta$  is governed by the difference equation of the form

$$\theta(k+1) = \theta(k). \quad (6.26)$$

Defining a new  $(n+m)$ -dimensional vector

$$\tilde{x}(k) = \begin{bmatrix} x(k) \\ \theta(k) \end{bmatrix}, \quad (6.27)$$

which is called the enlarged state vector, it follows from Eq. (6.1) that

$$\tilde{x}(k+1) = \tilde{f}(\tilde{x}(k), k) + \tilde{w}(k), \quad (6.28)$$

where  $\tilde{f}(\tilde{x}(k), k)$  is an  $(n+m)$ -dimensional vector-valued nonlinear function defined by

$$\hat{f}(\hat{x}(k), k) = \begin{bmatrix} f(x(k), \theta(k), k) \\ \theta(k) \end{bmatrix} \quad (6.29)$$

and  $\hat{w}(k)$  is an  $(n+m)$ -dimensional vector defined by

$$\hat{w}(k) = \begin{bmatrix} w(k) \\ 0 \end{bmatrix}. \quad (6.30)$$

It should be noted that even if the covariance matrix of  $w(k)$  is positive definite, the covariance matrix  $\hat{w}(k)$  is no longer positive definite.

On the other hand, from Eq. (6.2), the output observation can be rewritten in the form

$$z(k) = \hat{h}(\hat{x}(k), k) + v(k), \quad (6.31)$$

where  $\hat{h}(\hat{x}(k), k)$  is a  $p$ -dimensional vector-valued nonlinear function defined by

$$\hat{h}(\hat{x}(k), k) = h(x(k), k). \quad (6.32)$$

It should be noted that the system described by Eqs. (6.28) and (6.31) does not contain any unknown parameters, but the dimension of the enlarged state vector is increased by the dimension of the unknown parameter  $\theta$ . Thus, the parameter estimation problem can be converted to the state estimation problem for the system with enlarged state vector.

#### 6.4 Reformulation of the Problem

We shall reformulate the problem stated in section 6.2 from the viewpoint of the above discussion. For the convenience of description,



the tilde( $\sim$ ) is omitted in this and following sections, but there will be no confusions.

Consider a system in which the dynamical characteristic is expressed by

$$x(k+1) = f(x(k), k) + w(k), \quad (6.33)$$

where

$x(k)$  is an  $n$ -dimensional enlarged state vector,

$w(k)$  is an  $n$ -dimensional vector denoting the random disturbance,

$f(\cdot, \cdot)$  is an  $n$ -dimensional vector-valued nonlinear function.

The output observation is

$$z(k) = h(x(k), k) + v(k), \quad (6.34)$$

where

$z(k)$  is a  $p$ -dimensional vector denoting observation

$v(k)$  is a  $p$ -dimensional vector denoting the observation noise,

$h(\cdot, \cdot)$  is a  $p$ -dimensional vector-valued nonlinear function.

It should again be noted that the meaning of the above equations coincides with that of Eqs. (6.28) and (6.31); the  $x(k)$  contains both the original state vector and the unknown parameter. The  $w(k)$  and  $v(k)$  are assumed to be Gaussian white noises with means zero and covariance matrices  $Q(k)$  and  $R(k)$ , respectively, where  $Q(k)$  and  $R(k)$  are  $n \times n$  and  $p \times p$  symmetric, nonnegative definite matrices, respectively.

The problem is then reformulated as follows: on the basis of the observed data  $Z_k$ , find the best estimate  $\hat{x}(k)$  of the current state  $x(k)$ , minimizing the conditional expectation of the quadratic loss function

conditioned on  $Z_k$

$$E\{\|x(k) - \hat{x}(k)\|^2 | Z_k\}. \quad (6.35)$$

This problem is the same one treated in Chapter 4; therefore, the approximate estimation technique is readily applicable.

It is clear from the discussion in section 6.3 that, even if the unknown parameters are time-varying or randomly time-varying, it will be possible to embed the parameter estimation problem in the state estimation problem by rewriting Eq. (6.26) in the suitable form such as

$$\theta(k+1) = g(\theta(k), k) + \eta(k), \quad (6.36)$$

where  $\theta(k)$  is a Gaussian white random vector and  $g(\theta(k), k)$  is a vector-valued nonlinear function.

In Chapter 8, a different approach of estimating slowly time-varying parameters will be presented.

### 6.5 Approximate Nonlinear Estimator

As pointed out in section 4.3, it is almost impossible to find the optimal solution of the nonlinear estimation problem, we must be satisfied by an approximate solution. Since the derivation of the approximate solution of the nonlinear state estimation problem is shown in Chapter 4, we cite the final forms as the equations of the approximate estimator for the present problem.

Let  $x^*(k)$  be the approximate estimate of the state  $x(k)$  based upon observed data  $Z_k$ . By using Eqs. (4.40) to (4.45), we have the following recursive relations.

(i) The difference equation of the approximate estimator is

$$x^*(k+1) = f(x^*(k), k) + \Delta(k+1)[z(k+1) - h(f(x^*(k), k), k+1)], \quad (6.37)$$

where

$$\Delta(k+1) = P(k+1)H'(k+1)[H(k+1)P(k+1)H'(k+1) + R(k+1)]^{-1}. \quad (6.38)$$

It should be noted that  $\Delta(k+1)$  is the  $n \times p$  matrix which represents the gain of the updating term by the noisy observation.

(ii) The variance equations are given by

$$P(k+1) = F(k)M(k)F'(k) + Q(k) \quad (6.39)$$

and

$$M(k+1) = P(k+1) - \Delta(k+1)H(k+1)P(k+1). \quad (6.40)$$

(iii) The equivalent gain matrices are

$$F(k) = E\{[f(x(k), k) - f(x^*(k), k)][x(k) - x^*(k)]' | Z_k\} M^{-1}(k) \quad (6.41)$$

and

$$H(k+1) = E\{[h(x(k+1), k+1) - h(x^*(k+1|k), k+1)] \\ \times [x(k+1) - x^*(k+1|k)]' | Z_k\} P^{-1}(k+1), \quad (6.42)$$

where

$$x^*(k+1|k) = f(x^*(k), k). \quad (6.43)$$

The initial conditions to Eqs. (6.37) and (6.40) are  $x^*(0) = 0$ , and  $M(0) = E\{x(0)x'(0)\}$ .

The structure of the nonlinear estimator is completely determined by Eqs. (6.37) to (6.43). It should be noted that the resultant scheme does not contain the inverse matrices  $Q(k)$  or  $R(k+1)$ . This fact implies the assumption that both the covariance matrices are positive definite

is not necessary for the application of the above scheme to the actual problem. This plays an important role in the parameter estimation problem, because the covariance matrix of  $w(k)$  necessarily becomes singular from Eq. (6.30).

## 6.6 Digital Simulation Studies

Two examples are considered as the illustration of the present technique. The results of experimental simulation by a digital computer are also demonstrated for each examples.

### 6.6.1 Numerical Results for a First Order System

Consider the problem of estimating an unknown parameter for a simple first order system described by a scalar difference equation

$$x_1(k+1) = \alpha x_1(k) + \beta x_1^3(k) + w_1(k) \quad (6.44)$$

and let the output observation be

$$z(k) = x_1(k) + v(k), \quad (6.45)$$

where  $\alpha$  is an unknown constant to be estimated with a priori distribution  $N(0, \sigma^2)$ ,  $\beta$  is a known constant, and  $w_1(k)$  and  $v(k)$  are mutually independent Gaussian white noises with means zero and variances  $q$  and  $r$ , respectively.

According to Eq. (6.27), we define the enlarged state vector

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad (6.46)$$

where  $x_2(k) = \alpha$ . Then, the system to be considered can be expressed as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} x_1(k)x_2(k) + \beta x_1^3(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} w_1(k) \\ 0 \end{bmatrix} \quad (6.47)$$

and

$$z(k) = [1 \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + v(k). \quad (6.48)$$

Comparing with the general theory, we have

$$f(x(k), k) = \begin{bmatrix} x_1(k)x_2(k) + \beta x_1^3(k) \\ x_2(k) \end{bmatrix} \quad (6.49)$$

and

$$Q(k) = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}, \quad R(k) = r. \quad (6.50)$$

By using Eq. (6.41), the equivalent gain matrix  $F(k)$  becomes

$$F(k) = \begin{bmatrix} x_2^*(k) + 3\beta\{(x_1^*(k))^2 + M_{11}(k)\} & x_1^*(k) \\ 0 & 1 \end{bmatrix}. \quad (6.51)$$

By making use of Eqs. (6.37) to (6.40), we have the following approximate scheme.

(i) The difference equations of the estimator are

$$x_1^*(k+1) = \hat{\phi}(x^*(k)) + \Delta_1(k+1)[z(k+1) - \hat{\phi}(x^*(k))], \quad (6.52)$$

$$x_2^*(k+1) = x_2^*(k) + \Delta_2(k+1)[z(k+1) - \hat{\phi}(x^*(k))], \quad (6.53)$$

where

$$\Delta_1(k+1) = P_{11}(k+1)/[r + P_{11}(k+1)], \quad (6.54)_1$$

$$\Delta_2(k+1) = P_{12}(k+1)/[r + P_{11}(k+1)], \quad (6.54)_2$$

and

$$\gamma_{\phi}(x^*(k)) = x_1^*(k)x_2^*(k) + \beta[x_1^*(k)]^3. \quad (6.54)_3$$

(ii) The variance equations are

$$\begin{aligned} P_{11}(k+1) = & F_{11}^2(k)M_{11}(k) + 2F_{11}(k)F_{12}(k)M_{12}(k) \\ & + F_{22}^2(k)M_{22}(k) + q, \end{aligned} \quad (6.55)_1$$

$$P_{12}(k+1) = F_{11}(k)M_{12}(k) + F_{12}(k)M_{22}(k), \quad (6.55)_2$$

$$P_{22}(k+1) = M_{22}(k), \quad (6.55)_3$$

and

$$M_{11}(k+1) = rP_{11}(k+1)/[r + P_{11}(k+1)], \quad (6.56)_1$$

$$M_{12}(k+1) = rP_{12}(k+1)/[r + P_{11}(k+1)], \quad (6.56)_2$$

$$M_{22}(k+1) = P_{22}(k+1) - P_{12}^2(k+1)/[r + P_{11}(k+1)], \quad (6.56)_3$$

where  $P_{ij}$ ,  $M_{ij}$  and  $F_{ij}$  are elements of matrices  $P$ ,  $M$  and  $F$ , respectively.

Figs. 6.1 and 6.2 display the result of digital simulation for the numerical values:

$$\alpha = 0.9(\text{to be estimated}), \quad \beta = -0.01, \quad q = 0.09, \quad r = 0.64,$$

$$x_1^*(0) = x_2^*(0) = 0, \quad M_{11}(0) = 1.0, \quad M_{12}(0) = 0, \quad M_{22}(0) = 0.1.$$

For the reference, the result obtained under the assumption that no observation noise exists is also shown, and is denoted by  $\hat{\alpha}(k)$  and  $\Sigma_{\alpha}(k)$ . In this case, the a posteriori probability density function of  $\alpha$  conditioned on the data  $X_k = \{x(0), \dots, x(k)\}$  becomes

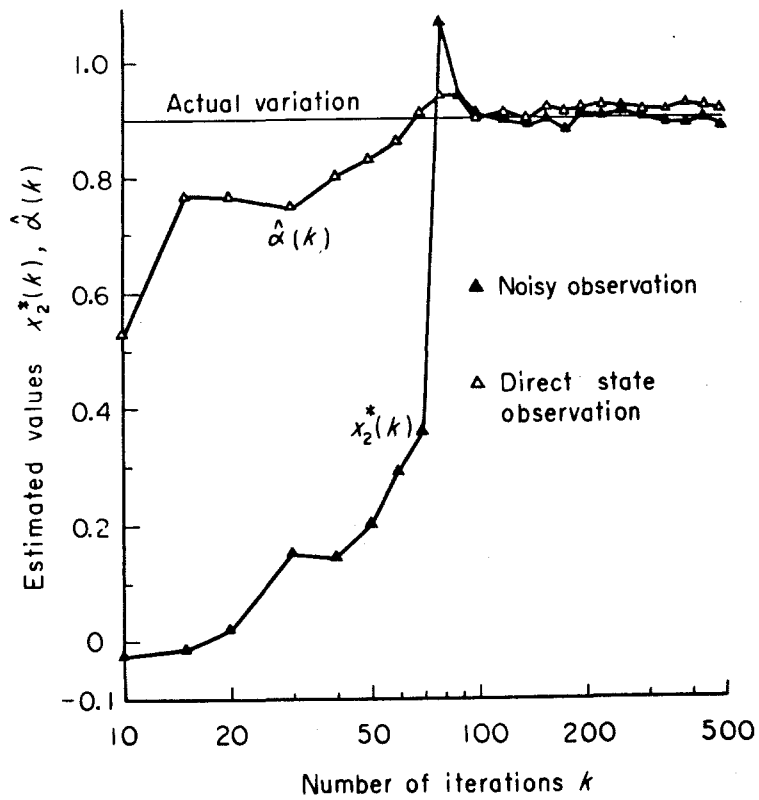


Fig. 6.1 Comparison of tracking behaviors for the unknown parameter  $\alpha$

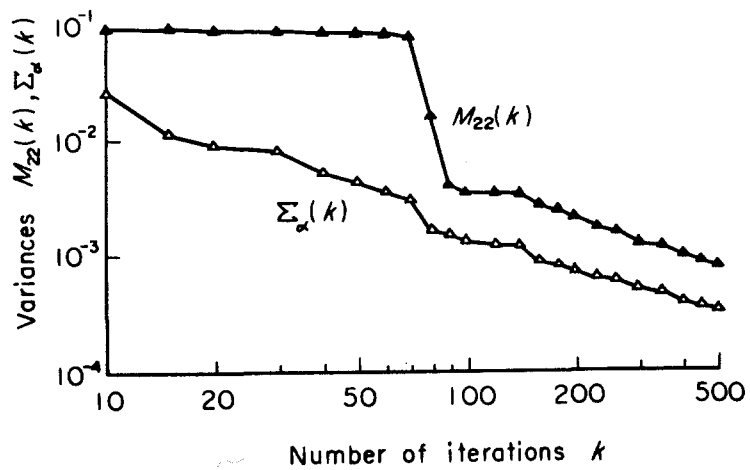


Fig. 6.2 Variation of variances of estimation errors  $M_{22}(k)$  and  $\Sigma_{\alpha}(k)$

$$p(\alpha | X_k) = C_8 \exp \left\{ -\frac{1}{2q} \sum_{j=0}^{k-1} [x_1(j+1) - \alpha x_1(j) - \beta x_1^3(j)]^2 - \frac{\alpha^2}{2\sigma^2} \right\}, \quad (6.57)$$

so that the best estimate of  $\alpha$  is given by

$$\hat{\alpha}(k) = \left\{ \frac{1}{q} \sum_{j=0}^{k-1} x_1(j) [x_1(j+1) - \beta x_1^3(j)] \right\} \Sigma_{\alpha}(k), \quad (6.58)_1$$

where  $\Sigma_{\alpha}(k)$  is the variance of the estimation error and is given by

$$\Sigma_{\alpha}(k) = \left\{ \frac{1}{q} \sum_{j=0}^{k-1} x_1^2(j) + \frac{1}{\sigma^2} \right\}^{-1}. \quad (6.58)_2$$

It should be noted that the pair  $\hat{\alpha}(k)$  and  $\Sigma_{\alpha}(k)$  is the optimal solution to the estimation problem when there is no observation noise.

Let us now consider the asymptotic behavior of the estimated value  $x_2^*(k)$  of the unknown parameter  $\alpha$ . It follows from Eqs. (6.55)<sub>3</sub> and (6.56)<sub>3</sub> that

$$M_{22}(k+1) - M_{22}(k) = -P_{12}^2(k+1)/[r + P_{11}(k+1)] \leq 0, \quad (6.59)$$

in addition,  $M_{22}(k)$  is nonnegative for all  $k$  by the definition, so that there exists a limiting value such that  $\lim_{k \rightarrow \infty} M_{22}(k) = M_{22} \geq 0$ . Therefore, application of Eqs. (6.55)<sub>2</sub>, (6.56)<sub>2</sub> and (6.59), taking account of the fact that  $F_{12}(k) = x_1^*(k)$  and that  $x_1^*(k)$  is not identically zero, yields

$$\lim_{k \rightarrow \infty} M_{22}(k) = \lim_{k \rightarrow \infty} E\{[x_2^*(k) - \alpha^*]^2 | Z_k\}. \quad (6.60)$$

This implies that the approximate estimate  $x_2^*(k)$  converges to  $\alpha^*$  in the mean square sense. But, because the present scheme is the approximate



one, it does not converge to  $\alpha$  (true value); that is, the expectation in Eq. (6.60) is computed by use of the approximate a posteriori probability density function which is assumed to be Gaussian. However, the result of experimental simulations shown in Figs. 6.1 and 6.2 may indicate  $\alpha^*$  is very close to  $\alpha$  (true value).

It should also be noted that  $\hat{\alpha}(k)$  given by Eq. (6.58)<sub>1</sub> converges to  $\alpha$  in the mean square sense, because the  $\Sigma_{\alpha}(k) \rightarrow 0$  as  $k \rightarrow \infty$ . This is guaranteed by the fact that the variance of the input noise  $w_1(k)$  is positive.

#### 6.6.2 Numerical Results for a Second Order System

As the second example, consider a system whose dynamical characteristic is described by the difference equations of the form

$$x_1(k+1) = x_1(k) + Tx_2(k), \quad (6.61)_1$$

$$x_2(k+1) = -\alpha x_1(k) - \beta x_1^3(k) + \gamma x_2(k) + \delta + w_2(k), \quad (6.61)_2$$

and let the output observation be

$$z(k) = x_1(k) + v(k), \quad (6.62)$$

where  $\alpha$  and  $\beta$  are unknown constants to be estimated, and  $\gamma$ ,  $\delta$  and  $T$  are known constants. The  $w_2(k)$  and  $v(k)$  are mutually independent Gaussian white noises with means zero and variances  $q$  and  $r$ , respectively.

Let us consider the problem of estimating  $\alpha$  and  $\beta$  on the basis of the noise corrupt data  $Z_k$ . Defining the enlarged state vector

$$x(k) = [x_1(k) \quad x_2(k) \quad x_3(k) \quad x_4(k)]', \quad (6.63)$$

where  $x_3(k) = \alpha$  and  $x_4(k) = \beta$ , the system to be considered becomes

$$x(k+1) = f(x(k), k) + w(k) \quad (6.64)$$

and

$$z(k) = [1 \ 0 \ 0 \ 0]x(k) + v(k), \quad (6.65)$$

where

$$f(x(k), k) = \begin{pmatrix} x_1(k) + Tx_2(k) \\ -x_1(k)x_3(k) - x_1^3(k)x_4(k) + \gamma x_2(k) + \delta \\ x_3(k) \\ x_4(k) \end{pmatrix} \quad (6.66)$$

and

$$w(k) = [0 \ w_2(k) \ 0 \ 0]^T. \quad (6.67)$$

The covariance matrices of  $w(k)$  and  $v(k)$  then become

$$Q(k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.68)$$

and

$$R(k) = r. \quad (6.69)$$

By making use of Eq. (6.41), the equivalent gain matrix  $F(k)$  becomes

$$F(k) = \begin{pmatrix} 1 & T & 0 & 0 \\ F_{21}(k) & F_{22}(k) & F_{23}(k) & F_{24}(k) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.70)$$

where

$$F_{21}(k) = -x_3^*(k) - 3[x_1^*(k)]^2 x_4^*(k) - 3x_4^*(k)M_{11}(k) - 6x_1^*(k)M_{14}(k), \quad (6.71)_1$$

$$F_{22}(k) = \gamma, \quad (6.71)_2$$

$$F_{23}(k) = -x_1^*(k), \quad (6.71)_3$$

$$F_{24}(k) = -[x_1^*(k)]^3 - 3x_1^*(k)M_{11}(k). \quad (6.71)_4$$

Therefore, the recursive relations which characterize the structure of the approximate estimator become as follows.

(i) The difference equation of the estimator is

$$x^*(k+1) = f(x^*(k), k) + \Delta(k+1)[z(k+1) - f_1(x^*(k), k)], \quad (6.72)$$

where  $\Delta(k+1)$  is a  $4 \times 1$  gain matrix defined by

$$\Delta_{i1}(k+1) = P_{i1}(k+1)/[r + P_{11}(k+1)], \quad i = 1 \sim 4. \quad (6.73)$$

and

$$f_1(x^*(k), k) = x_1^*(k) + Tx_2^*(k). \quad (6.74)$$

(ii) The variance equations are

$$P(k+1) = F(k)M(k)F'(k) + Q(k), \quad (6.75)$$

where  $P(k+1)$  is a  $4 \times 4$  matrix, and

$$M_{ij}(k+1) = P_{ij}(k+1) - P_{1i}(k+1)P_{1j}(k+1)/[r + P_{11}(k+1)], \quad (6.76)$$

for  $i, j = 1 \sim 4$ .

Digital simulation studies are also carried out by using pseudo-random numbers for this example, and the results are shown in Figs. 6.3 through 6.6.

Figs. 6.3 and 6.4 display the result for the numerical values:

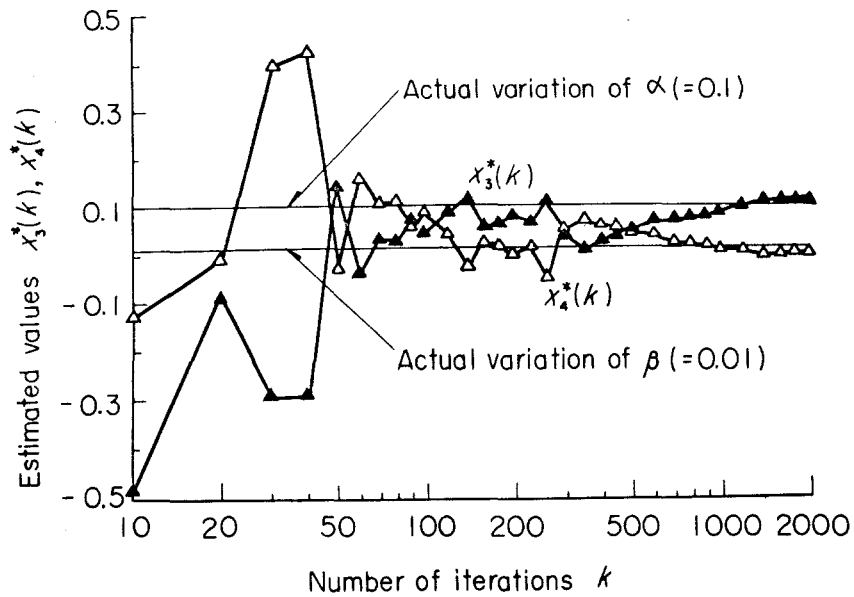


Fig. 6.3 Tracking behavior for unknown parameters  $\alpha$  and  $\beta$  when input bias  $\delta$  is zero

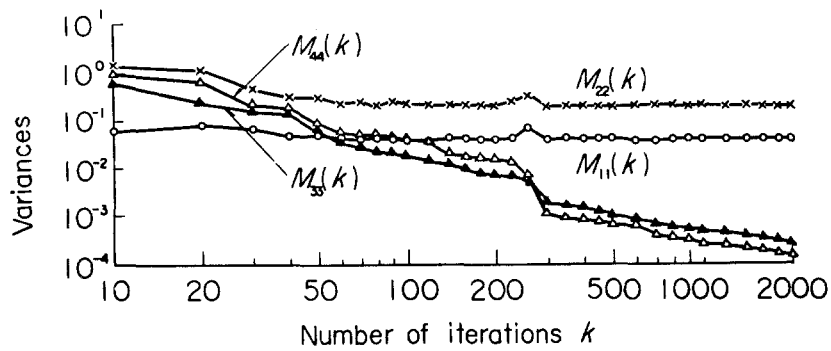


Fig. 6.4 Variation of variances of estimation errors

$\alpha = 0.1, \quad \beta = 0.01, \quad \gamma = 0.95, \quad \delta = 0, \quad T = 0.1, \quad q = 0.04,$   
 $r = 0.16, \quad x(0) = 0, \quad x^*(0) = 0, \quad M(0) = I \text{ (unit matrix).}$

The  $x_3^*(k)$  and  $x_4^*(k)$  represent the estimated values of the unknown parameters  $\alpha$  and  $\beta$ , respectively, and  $M_{jj}(k)$ 's represent the variances

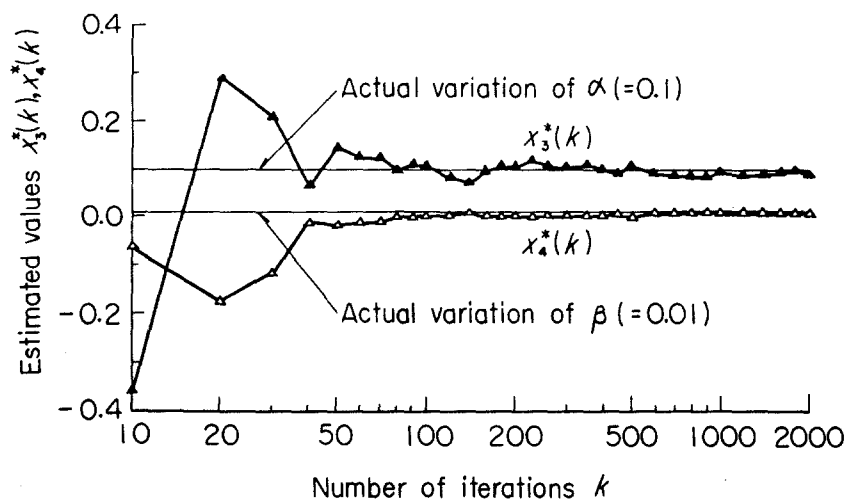


Fig. 6.5 Tracking behavior for unknown parameters  $\alpha$  and  $\beta$  when input bias  $\delta$  is 0.2

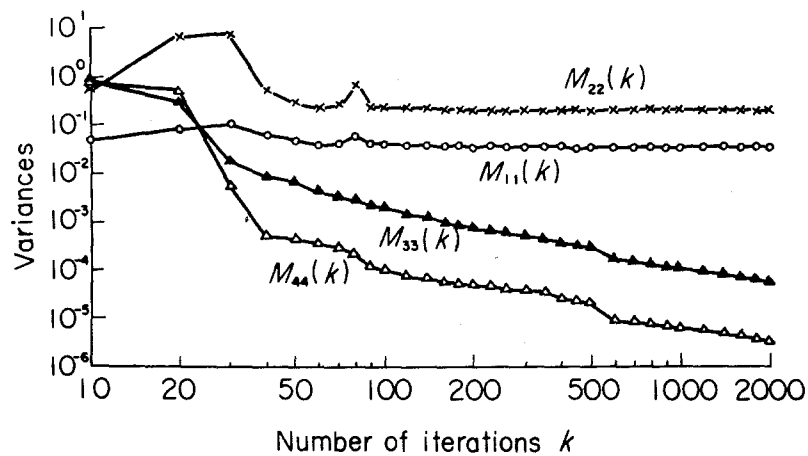


Fig. 6.6 Variation of variances of estimation errors

of estimation errors of  $x_j(k)$ 's.

Figs. 6.5 and 6.6 display the result for the same numerical values and initial conditions, except for the input bias  $\delta = 0.2$ .

Figs. 6.4 and 6.6 show that the variances  $M_{11}(k)$  and  $M_{22}(k)$  are larger than  $M_{33}(k)$  and  $M_{44}(k)$ ; this fact implies that the parameter estimation is sometimes satisfactory even if the state estimation is rather rough. It is also shown for this example that  $\lim_{k \rightarrow \infty} M_{33}(k) = 0$  and  $\lim_{k \rightarrow \infty} M_{44}(k) = 0$  after some manipulations. The comparison of these figures shows that the insertion of known input signal (or test signal) reduces the parameter estimation errors, but not the state estimation errors.

#### 6.7 Concluding Remarks

In this chapter, the problem has considered of estimating unknown parameters for noisy nonlinear dynamical systems. The discussion concerning the parameter estimation is presented from the Bayesian point of view, and it has been shown that the parameter estimation can not be treated separately from the state estimation problem when both the input and output data are corrupted by noises. From this point of view, the parameter estimation problem is successfully embedded in the state estimation problem by regarding unknown parameters as another state variables. The approximate state estimation technique developed in Chapter 4 is then extended to this situation.

Two examples have been presented as the illustration of the technique, including digital simulation studies. The results of simulation studies and the fact that the estimated values of unknown parameters converge to some definite values, may indicate the feasibility of the present scheme.

## CHAPTER 7

### ACCURACY CONSIDERATION FOR APPROXIMATE PARAMETER ESTIMATION TECHNIQUE

#### 7.1 Introduction and Problem Statement

In Chapter 6, we have developed a parameter estimation technique for nonlinear dynamical systems by extending the approximate state estimation technique presented in Chapter 4. Moreover, a comparative study has also been made for a first order nonlinear system in section 6.6.1.

In this chapter, we consider the accuracy of the parameter estimation technique by the numerical evaluation of the a posteriori probability density function of the unknown parameter. The approach of this chapter is the same as that of Chapter 5.

Let us consider a first order system described by the linear

difference equation of the form

$$x(k+1) = \alpha x(k) + w(k), \quad (7.1)$$

where  $x(k)$  is a state variable,  $w(k)$  is a white noise with  $N(0, q)$ , and  $\alpha$  is an unknown parameter to be estimated. Let the output observation be the state variable corrupted by an additive noise, i.e.,

$$z(k) = x(k) + v(k), \quad (7.2)$$

where  $z(k)$  is the output observation, and  $v(k)$  is a white noise with  $N(0, r)$  and is independent of the  $w(k)$ .

## 7.2 Approximate Parameter Estimation Scheme

Since the present system is the same as considered in the example in section 6.6.1, the approximate estimation scheme is already given there, where  $\beta = 0$ .

(i) The difference equations of the estimator are

$$x^*(k+1) = \alpha^*(k)x^*(k) + \Delta_1(k+1)[z(k+1) - \alpha^*(k)x^*(k)] \quad (7.3)$$

$$\alpha^*(k+1) = \alpha^*(k) + \Delta_2(k+1)[z(k+1) - \alpha^*(k)x^*(k)], \quad (7.4)$$

where

$$\Delta_1(k+1) = P_{11}(k+1)/[r + P_{11}(k+1)], \quad (7.5)$$

$$\Delta_2(k+1) = P_{12}(k+1)/[r + P_{11}(k+1)], \quad (7.6)$$

and where  $x^*(k)$  and  $\alpha^*(k)$  are the estimates of  $x(k)$  and  $\alpha(k)$  based on the data  $Z_k$ , respectively.

(ii) The variance equations are



$$P_{11}(k+1) = [\alpha^*(k)]^2 M_{11}(k) + 2[\alpha^*(k)x^*(k)]M_{12}(k) + [x^*(k)]^2 M_{22}(k) + q, \quad (7.7)_1$$

$$P_{12}(k+1) = \alpha^*(k)M_{12}(k) + x^*(k)M_{22}(k), \quad (7.7)_2$$

$$P_{22}(k+1) = M_{22}(k), \quad (7.7)_3$$

and

$$M_{11}(k+1) = rP_{11}(k+1)/[r + P_{11}(k+1)], \quad (7.8)_1$$

$$M_{12}(k+1) = rP_{12}(k+1)/[r + P_{11}(k+1)], \quad (7.8)_2$$

$$M_{22}(k+1) = P_{22}(k+1) - P_{12}^2(k+1)/[r + P_{11}(k+1)], \quad (7.8)_3$$

where  $P_{ij}$  and  $M_{ij}$  are elements of matrices  $P$  and  $M$ , respectively. The indices 1 and 2 represent  $x$  and  $\alpha$ , respectively.

(iii) The equivalent gain matrix is given by

$$F(k) = \begin{pmatrix} \alpha^*(k) & x^*(k) \\ 0 & 1 \end{pmatrix}. \quad (7.9)$$

### 7.3 Exact Expression of the Joint a Posteriori Probability Density Function

By using Eq. (6.25), the recurrence relation for the joint a posteriori probability density function of  $x(k)$  and  $\alpha$  becomes

$$p(x(k+1), \alpha | Z_{k+1}) = C_1 \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2r}[z(k+1) - x(k+1)]^2 - \frac{1}{2q}[x(k+1) - \alpha x(k)]^2\right\} p(x(k), \alpha | Z_k) dx(k), \quad (7.10)$$

where  $C_1$  is a normalizing factor independent of  $x(k+1)$  and  $\alpha$ . The

present situation is the simplest case of parameter estimation in the presence of both the input and output noises. However, the successive evaluation of the a posteriori probability density function is almost impossible; so that the exact solution of this problem can not be obtained†.

What we can do at this stage is to evaluate the a posteriori probability density function by a digital computer and to obtain the numerical solution to this problem; and we can compare the result with the approximate solution obtainable by the method of section 7.2.

For the numerical computation, it is assumed that the  $p(x(k), \alpha | Z_k)$  is distributed over the domain  $D$ , where  $D = [a, b] \times [c, d]$ , neglecting the contribution in the outside of the domain (see Fig. 7.1).

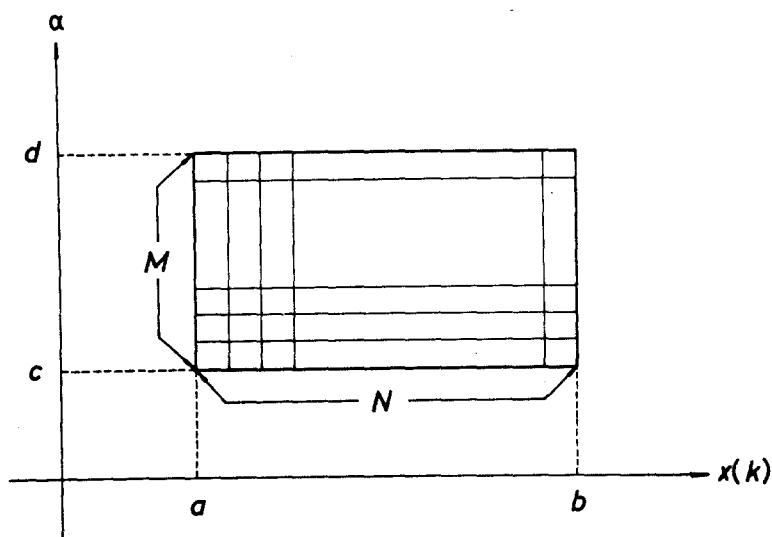


Fig. 7.1 Domain of integration

---

† In Eq. (6.19), the a posteriori probability density function of unknown parameter  $\alpha$  conditioned on observation  $z(1)$  is evaluated.

Furthermore, the domain  $D$  is partitioned into  $N \times M$  small areas, and Eq. (7.10) is approximated as follows.

$$p(x_i, \alpha_j | Z_{k+1}) \approx C_2 \exp\left\{-\frac{1}{2r}[z(k+1) - x_i]^2\right\} \\ \times \sum_{\ell=0}^N \exp\left\{-\frac{1}{2q}[x_i - \alpha_j \xi_\ell]^2\right\} p(\xi_\ell, \alpha_j | Z_k), \quad (7.11)$$

where  $i = 0, 1, \dots, N$  and  $j = 0, 1, \dots, M$ , and  $x_i$ ,  $\alpha_j$  and  $\xi_\ell$  are all dummy variables, and  $C_2$  is a normalizing factor.

#### 7.4 Digital Simulation Studies

The numerical values used in the computation are as follows.

$$\alpha = 0.9, \quad q = 0.09, \quad r = 0.16, \quad x(0) = 0,$$

$$D = [-1.5, 1.5] \times [-1, 2], \quad N = 60, \quad M = 60,$$

$$x^*(0) = 0.5, \quad \alpha^*(0) = 0.5, \quad M(0) = \begin{pmatrix} 0.64 & 0 \\ 0 & 0.25 \end{pmatrix}$$

$p(x(0), \alpha | Z_0)$  is a 2-dimensional Gaussian with  $\hat{x}(0) = 0.5$ ,  $\hat{\alpha}(0) = 0.5$ ,

$$\sigma_x^2(0) = 0.64, \quad \sigma_{\alpha x}(0) = \sigma_{x\alpha}(0) = 0, \quad \sigma_\alpha^2(0) = 0.25.$$

By using pseudorandom numbers generated by a digital computer, the data  $Z_{20} = \{z(1), \dots, z(20)\}$  are formed. The data are then processed by using two different procedures explained in sections 7.2 and 7.3.

Fig. 7.2 displays the comparison of the estimated values  $\hat{x}(k)$  and  $x^*(k)$  of the state variable  $x(k)$ , where  $\hat{x}(k)$  is the a posteriori conditional mean and  $x^*(k)$  is the approximate estimate. Fig. 7.3 displays the transition of the a posteriori probability density functions

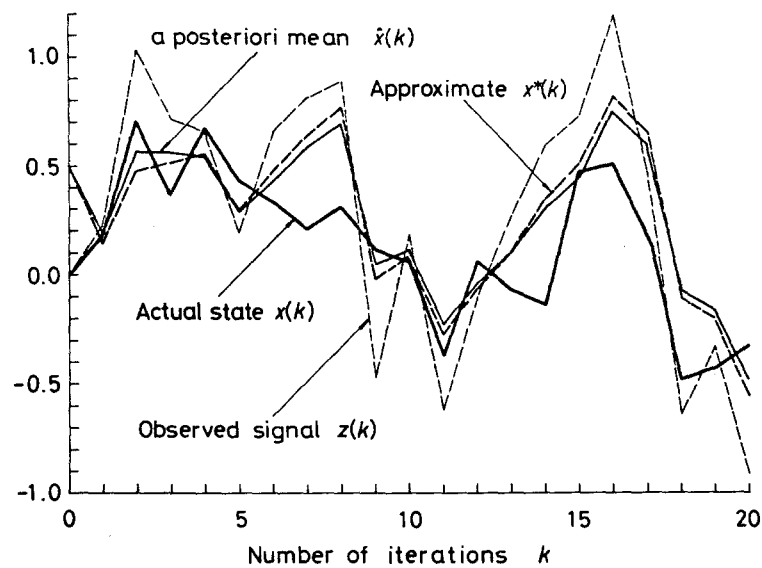


Fig. 7.2 Comparison of the estimated values  $\hat{x}(k)$  and  $x^*(k)$  of the state  $x(k)$

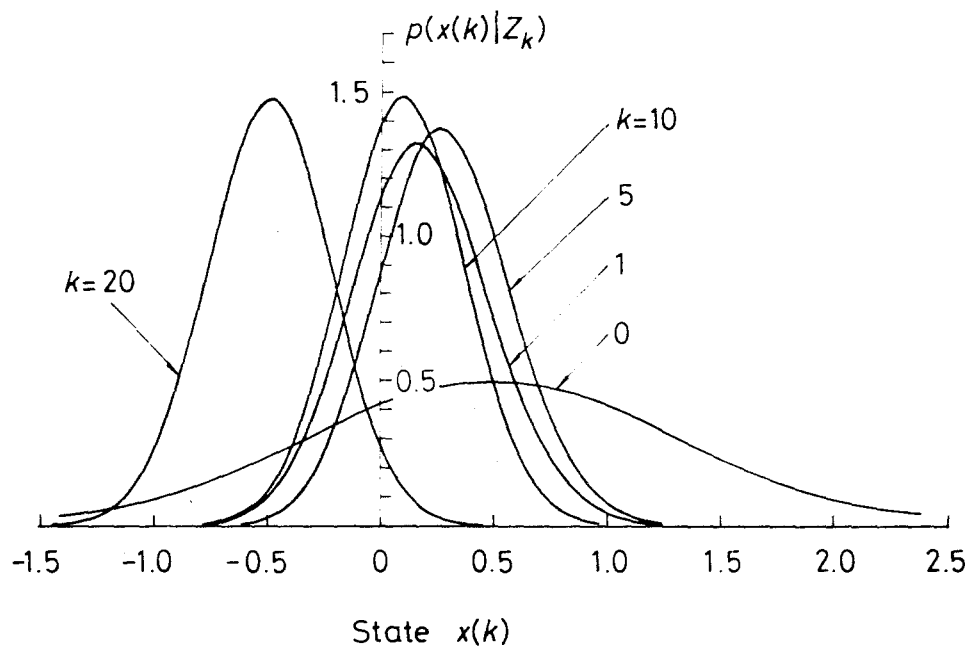


Fig. 7.3 Transition of the a posteriori probability density functions of  $x(k)$  conditioned on  $Z_k$

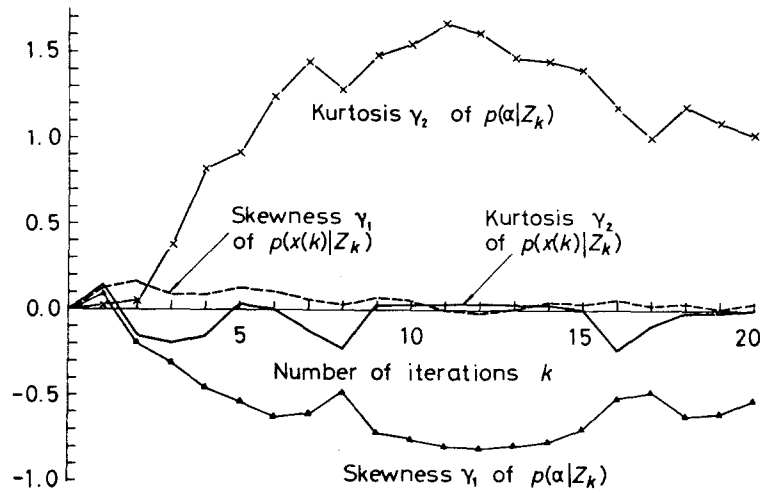


Fig. 7.4 Skewnesses and kurtosises of the a posteriori probability densities  $p(x(k)|Z_k)$  and  $p(\alpha|Z_k)$

$p(x(k)|Z_k)$  of  $x(k)$  conditioned on  $Z_k$ . The skewness  $\gamma_1$  and kurtosis  $\gamma_2$  of the  $p(x(k)|Z_k)$  are shown in Fig. 7.4.

Fig. 7.5 displays the transition of the a posteriori probability density functions  $p(\alpha|Z_k)$  of  $\alpha$  conditioned on  $Z_k$ , and Fig. 7.6 displays the comparison of the estimated values  $\hat{\alpha}(k)$  and  $\alpha^*(k)$  of  $\alpha$  by the two different procedures. The skewness and kurtosis of the  $p(\alpha|Z_k)$  are also shown in Fig. 7.4. These figures show that the  $p(x(k)|Z_k)$  is nearly Gaussian and that the  $p(\alpha|Z_k)$  slightly deviates from a Gaussian distribution; the skewness and kurtosis of the  $p(\alpha|Z_k)$  is rather large compared with those of the  $p(x(k)|Z_k)$ . In this example, the approximate estimate  $\alpha^*(k)$  shows good behavior compared with the a posteriori conditional mean  $\hat{\alpha}(k)$ .

Table 7.1 shows the covariances of the a posteriori probability

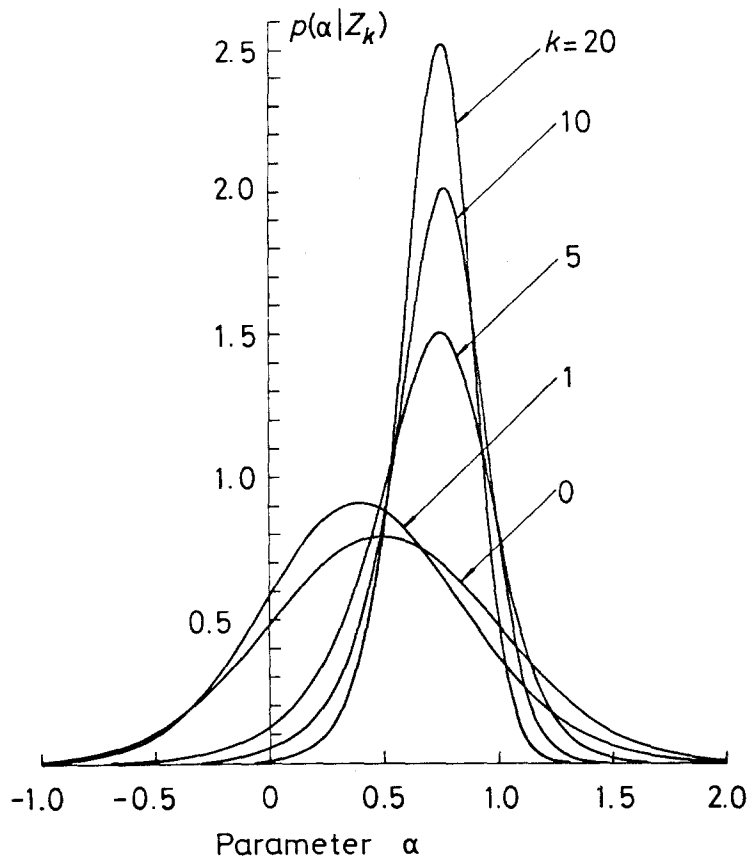


Fig. 7.5 Transition of the a posteriori probability density functions of  $\alpha$  conditioned on the data  $Z_k$

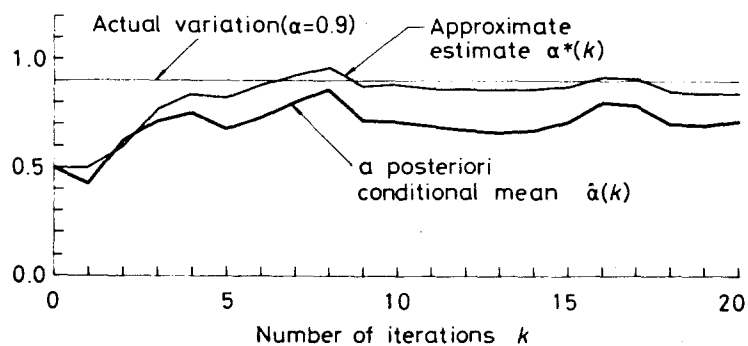


Fig. 7.6 Comparison of the estimated values  $\hat{\alpha}(k)$  and  $\alpha^*(k)$  of the unknown parameter  $\alpha$

k	$\sigma_x^2(k)$	$M_{11}(k)$	$\sigma_\alpha^2(k)$	$M_{22}(k)$	$\sigma_{x\alpha}(k)$	$M_{12}(k)$
0	0.640000	0.640000	0.250000	0.250000	0.000000	0.000000
1	0.095281	0.097561	0.197054	0.250000	0.032578	0.000000
2	0.094521	0.068366	0.205994	0.245540	0.061650	0.020216
3	0.101990	0.085524	0.142175	0.195760	0.063929	0.060889
4	0.097587	0.095837	0.104932	0.141277	0.049870	0.059125
5	0.085023	0.098477	0.089563	0.102083	0.034189	0.049106
6	0.084492	0.086893	0.079527	0.087769	0.029321	0.032348
7	0.088594	0.089740	0.062475	0.074262	0.029755	0.030806
8	0.089253	0.095138	0.046348	0.059557	0.026575	0.030883
9	0.077604	0.099022	0.051645	0.045887	0.017011	0.028871
10	0.073175	0.081499	0.051699	0.043982	0.007548	0.012231
11	0.071207	0.079129	0.052581	0.043287	-0.001416	0.007495
12	0.071073	0.077277	0.054057	0.043205	-0.005164	-0.002602
13	0.070202	0.077066	0.054485	0.053109	0.000653	-0.002820
14	0.071952	0.076735	0.052180	0.043097	0.008546	0.000986
15	0.076606	0.078185	0.045655	0.042353	0.015220	0.007803
16	0.081068	0.082060	0.034449	0.039994	0.018971	0.013817
17	0.084138	0.090483	0.029347	0.034183	0.018120	0.019942
18	0.073596	0.089960	0.032380	0.029585	0.009314	0.017945
19	0.070655	0.078229	0.031961	0.029086	-0.000240	0.006392
20	0.073490	0.076151	0.029396	0.029084	-0.008008	-0.000307

Table 7.1 Comparison of covariances of the a posteriori probability density function  $p(x(k), \alpha | Z_k)$  and the approximate covariances  $M_{ij}(k)$

density function  $p(x(k), \alpha | Z_k)$  and the approximate covariances  $M_{ij}(k)$ . The  $M_{11}(k)$  and  $\sigma_x^2(k)$  show good agreement, and  $M_{22}(k)$  and  $\sigma_\alpha^2(k)$  also show good agreement; however,  $M_{12}(k)$  and  $\sigma_{x\alpha}(k)$  are rather different each other. This fact may cause the difference of the two estimates

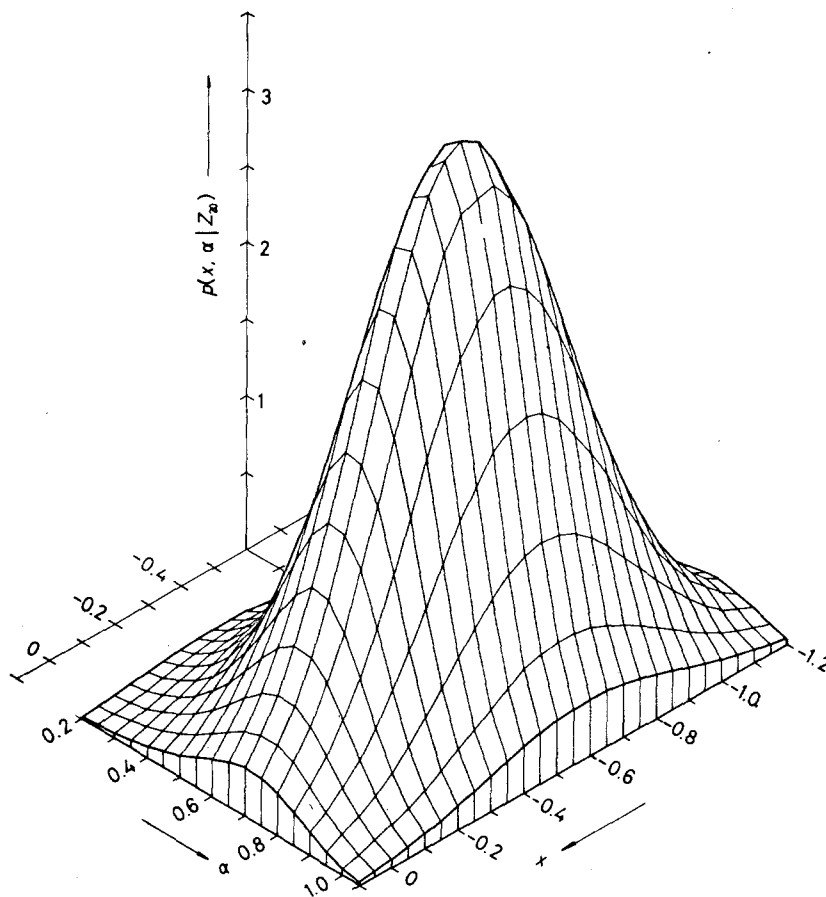


Fig. 7.7 Schematic view of the joint a posteriori probability density function of  $x(20)$  and  $\alpha$  conditioned on the data  $Z_{20}$

$\hat{\alpha}(k)$  and  $\alpha^*(k)$  of unknown parameter  $\alpha$ , as shown in Fig. 7.6.

Fig. 7.7 displays the 2-dimensional joint a posteriori probability density function of  $x(20)$  and  $\alpha$  conditioned on the twenty data  $Z_{20}$ .

### 7.5 Concluding Remarks

In this chapter, we have demonstrated a comparative study on the



parameter estimation problem for a simple first order system with an unknown parameter. This example is only a special case of the parameter estimation problem, simulated by using a set of particular pseudorandom numbers. Therefore, it is impossible to draw definitive conclusions on the accuracy of the approximate state estimation technique developed in Chapter 6. The results, which can be drawn from the simulation studies, are as follows.

The assumption, that the joint a posteriori probability density function of  $x(k)$  and  $\alpha$  conditioned on  $Z_k$  is nearly Gaussian, is satisfied in this example. The approximate estimate  $x^*(k)$  of  $x(k)$  shows good agreement with the conditional mean  $\hat{x}(k)$  of  $x(k)$  conditioned on  $Z_k$ . However, the approximate estimate  $\alpha^*(k)$  of the unknown parameter is shown to be rather different from the a posteriori conditional mean  $\hat{\alpha}(k)$ . This is partly because the method of the evaluation of the joint a posteriori probability density function is rather rough; that is, the mesh may not be small enough to give a good approximation of the true a posteriori probability density function; and partly because the realization of pseudorandom numbers by a digital computer may be bad.

The approach used in Chapters 5 and 7 mainly depends upon the speed of digital computer available. Therefore, it is necessary to develop a more simple procedure to estimate the accuracy of the approximate state and parameter estimation technique presented in Chapters 4 and 6. This problem, however, is not treated in this dissertation.

## CHAPTER 8

### ESTIMATION METHOD OF SLOWLY TIME-VARYING PARAMETERS AND ITS APPLICATION TO ADAPTIVE CONTROL

#### 8.1 Introduction

It is well-known that if the system to be controlled is linear and deterministic, the control law which minimizes a quadratic loss function is a linear function of the state variables. A similar result is also derived for stochastic systems; under noisy observation, if the controlled system is linear and is subjected to additive white Gaussian noise, the control law which minimizes the expectation of a quadratic loss function is a linear function of the optimal estimates of state variables based upon the available data. This fact is known as the "certainty equivalence principle"[15, 20]†.

---

† Recently, Wonham[56] derived a more general "separation theorem" for linear systems with non-quadratic error criteria.

In physical processes, however, we may encounter the cases where the controlled system contains unknown parameters, which may or may not be constant, and is subjected to random disturbances. In such cases, it is inevitable to estimate unknown parameters as precisely as possible so that the control law can be modified to maintain a good performance. This situation implies that the adaptive control is most desirable to compensate the deterioration of the control performance due to the existence of unknown parameters and their unpredictable changes[5].

Even if the controlled system is linear with respect to the state variables, and if the system contains unknown parameters, then the system behaves like a nonlinear system with respect to the enlarged state variables. This fact is clearly seen from Eqs. (7.1) and (7.3). In this case, therefore, the certainty equivalence principle does not hold, so that it is almost hopeless to find the control law which minimizes a specified criterion function together with optimizing the estimation of state variables and unknown parameters[1].

This chapter presents an approximation method of attacking the problem mentioned above, and is divided into two parts[42].

The first part is devoted to the development of a sequential estimation technique for slowly time-varying parameters by extending the result in Chapter 6. In the remainder, the resultant scheme is applied to the adaptive compensation of the deterioration of the control performance due to unknown parameters in the controlled systems.

## 8.2 Statement of Problem

First, we consider the problem of estimating slowly time-varying parameters for nonlinear dynamical systems. Consider a controlled system described by the vector difference equation

$$x(k+1) = f(x(k), \theta(k), k) + u(k) + w(k), \quad (8.1)$$

where

$x(k)$  is an  $n$ -dimensional state vector,

$u(k)$  is an  $n$ -dimensional control vector,

$w(k)$  is an  $n$ -dimensional vector denoting the random disturbance,

$\theta(k)$  is an  $m$ -dimensional unknown parameter vector, slowly time-varying

$f$  is an  $n$ -dimensional vector-valued nonlinear function.

It is assumed that the  $\theta(k)$  has a priori distribution, Gaussian with mean  $\theta^*(0)$  and covariance matrix  $V(0)$ , and that the initial state  $x(0)$  is an independent random variable with  $N(x^*(0), M(0))$ . Let the output observation be made on the process  $z(k)$  in the form

$$z(k) = h(x(k), k) + v(k), \quad (8.2)$$

where

$z(k)$  is a  $p$ -dimensional vector denoting the output observation,

$v(k)$  is a  $p$ -dimensional vector denoting the observation noise,

$h$  is a  $p$ -dimensional vector-valued nonlinear function.

It is also assumed that the  $w(k)$  and  $v(k)$  are mutually independent Gaussian white noises with means zero and covariance matrices  $Q(k)$  and  $R(k)$ , respectively, where  $Q(k)$  and  $R(k)$  are  $n \times n$  and  $p \times p$  symmetric and nonnegative definite matrices, respectively.

The problem to be considered is to estimate the unknown parameter vector  $\theta(k)$  on the basis of the available data  $Z_k = \{z(1), \dots, z(k)\}$ , where the variation of  $\theta(k)$  is unknown but is assumed to be slow; term "slow" means that the percentage of parameter changes is small during the successive sampling instants.

There is no general way of estimating unknown parameters in this situation. In this chapter, therefore, we generalize the approximate parameter estimation method to cover the estimation of slowly time-varying parameters. The principal approach is to modify the error covariance matrix by introducing the threshold values so that the greater weight is assigned to the latest observation.

The main idea cited are the method of weighted least square by Kalman[22], and the perturbation method in adaptive control systems[2].

### 8.3 Estimation Method for Time-Varying Parameters

From the viewpoint of sequential estimation, the most important factor of the structure of the nonlinear estimator developed in Chapter 6 is the gain matrix  $\Delta(k+1)$ . This gain matrix is automatically determined by Eq. (6.38). In the determination of  $\Delta(k+1)$ , the error covariance matrix  $M(k+1)$  plays a significant role, because, from Eq. (4.39)<sub>1</sub>, if  $R(k+1)$  is nonsingular, the gain matrix is expressed as

$$\Delta(k+1) = M(k+1)H'(k+1)R^{-1}(k+1). \quad (8.3)$$

If a certain unknown parameter, say  $x_v(k)$ , is constant, the variance  $M_{vv}(k)$  of the estimation error of  $x_v(k)$  monotonously decreases. In fact, since, from Eqs. (6.39) and (6.40), we have  $P_{vv}(k+1) = M_{vv}(k)$ ,

it follows that

$$M_{vv}(k+1) = M_{vv}(k) - \{P(k+1)H'(k+1)[H(k+1)P(k+1)H'(k+1) + R(k+1)]^{-1} \times H(k+1)P(k+1)\}_{vv}. \quad (8.4)^\dagger$$

Since  $P(k+1)$  is symmetric and positive definite, and since  $R(k+1)$  is symmetric and nonnegative definite, the  $n \times n$  matrix  $PH'[HPH' + R]^{-1}$  is positive definite. Therefore, the  $(v, v)$ -component of this matrix becomes positive. This fact implies that  $M_{vv}(k+1) < M_{vv}(k)$ , that is,  $M_{vv}(k)$  is a monotone decreasing sequence. Moreover,  $M_{vv}(k) \geq 0$  by the definition, so that  $M_{vv}(k)$  has a limiting value.

Now consider in detail a particular case where only one output observation

$$z(k) = [h \ 0 \ \dots \ 0]x(k) + v(k) \quad (8.5)$$

is available, and assume that  $R(k) = r = \text{const.}$  In this case, it follows from Eqs. (6.37), (6.38) and (6.40) that

$$x_v^*(k+1) = x_v^*(k) + \Delta_{v1}(k+1)[z(k+1) - hx_1^*(k)], \quad (8.6)$$

where

$$\Delta_{v1}(k+1) = hP_{11}(k+1)/[r + P_{11}(k+1)] \quad (8.7)$$

and

$$M_{vv}(k+1) = M_{vv}(k) - h^2 P_{v1}^2(k+1)/[r + P_{11}(k+1)]. \quad (8.8)$$

Since, as mentioned above,  $M_{vv}(k)$  has a limiting value, it follows from Eq. (8.8) that  $\lim_{k \rightarrow \infty} P_{v1}(k) = 0$ . Therefore, from Eq. (8.7), we have

---

<sup>†</sup> See, for example, Eqs. (6.55)<sub>3</sub> and (6.56)<sub>3</sub>.

$\lim_{k \rightarrow \infty} \Delta_{v1}(k) = 0$ . This fact is really desirable when the unknown parameter is constant†.

The above consideration suggests that even if the unknown parameter is slowly time-varying, it is possible to estimate  $x_v(k)$  by a suitable modification of the gain  $\Delta_{v1}(k+1)$  given by Eq. (8.7). Roughly speaking, it will be possible to estimate slowly time-varying parameters by assigning a greater weight to the latest observation. This idea is somewhat similar to the weighted least square method by Kalman[22].

Since  $\Delta_{v1}(k)$  takes both positive and negative values, the direct modification of this gain is impossible. Then we modify the variance  $M_{vv}(k)$ , which is closely related to the gain  $\Delta_{v1}(k)$  as shown in Eq. (8.3). Because the variance  $M_{vv}(k)$  is always nonnegative, it is possible to assign any nonnegative value to this variance, keeping the covariance matrix  $M(k)$  positive definite.

Let  $x_v(k)$  be slowly time-varying parameters in question,  $\epsilon_v$  be the preassigned nonnegative constants, and  $N_v$  be the first time instant such that  $M_{vv}(k) \leq \epsilon_v$ , where  $v = \alpha_1, \dots, \alpha_m$ .

The modification method of the covariance matrix that covers the problem of estimating slowly time-varying parameter is stated as follows.

---

† In the method of stochastic approximation, the weight, say  $\gamma_k$ , of the updating term by the noisy observation must satisfy the condition that

$$\sum \gamma_k = \infty \quad \text{and} \quad \sum \gamma_k^2 < \infty,$$

in order to ensure the convergence of the scheme. Therefore,  $\gamma_k \rightarrow 0$  as  $k \rightarrow \infty$ . See Dvoretzky[13].

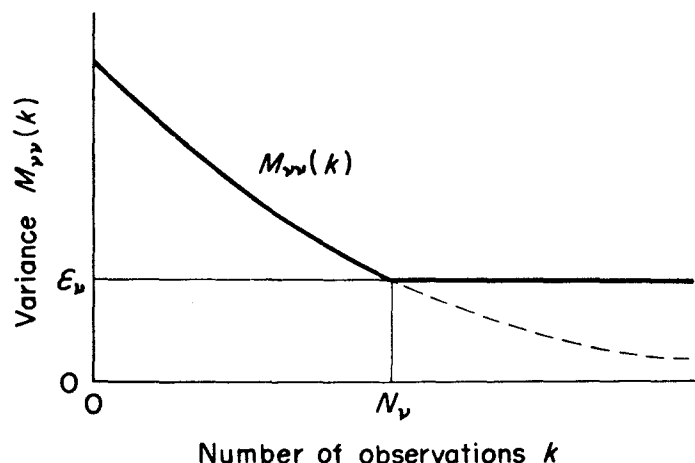


Fig. 8.1 A typical example of the modification of the variance  $M_{vv}(k)$  of a slowly time-varying parameter

- (i) No modification is given for  $0 \leq k < N_v$ .
- (ii) Let  $M_{vv}(k) = \epsilon_v$ † for  $k \geq N_v$ , where  $v = \alpha_1, \dots, \alpha_m$ .

Fig. 8.1 shows the view of the modification scheme presented here. The modification  $M_{vv}(k) = \epsilon_v$ , for  $k \geq N_v$ , can be regarded as the perturbation signal in adaptive control systems[2]. The determination of the threshold values  $\epsilon_v$ ,  $v = \alpha_1, \dots, \alpha_m$ , belongs to numerical experiment††.

Thus, the estimation method of slowly time-varying parameters

---

† It is possible, at least formally, to take  $\epsilon_v$  as some function of time  $k$ , but the realization is very difficult. Although different methods of the modification of  $M_{vv}(k)$  are of course possible, the suggested one is promising.

†† In the method of stochastic approximation, if the weight  $\gamma_k$  of the updating term is equal to  $A/k$  ( $A > 0$ ), then the condition to ensure the convergence is satisfied. However, if the number of observations is finite, the value  $A$  actually influences the rapidity of convergence. See reference [6].



simply becomes the combination of the parameter estimation scheme given by Eqs. (6.37) to (6.42) with the modification of the covariance matrix  $M(k)$ .

#### 8.4 Application to Adaptive Control in the Presence of Noise

An adaptive control system can be considered as a system which modifies its control law to keep a good performance on the basis of the information acquired from its experience. From this point of view, the technique of parameter estimation developed in section 8.3 can be applied to the adaptive compensation of the deterioration of the control performance due to parameter variations in the controlled system, under the random environment. The resulting scheme of the adaptive control system is shown in Fig. 8.2. In this figure, the configuration of the controller is determined on the basis of the a priori information about the controlled system. During the normal operation, the parameter variations in the plant are estimated and are fed back to the controller together

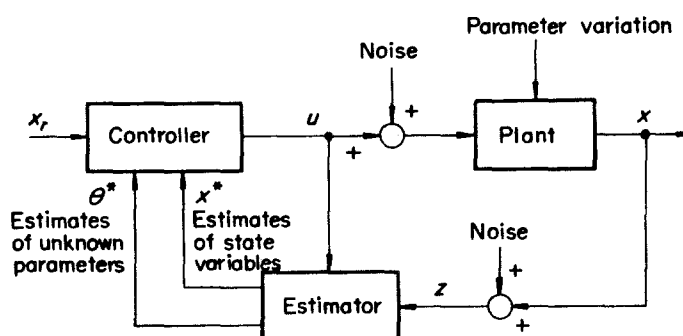


Fig. 8.2 Resulting configuration of the adaptive control system having the scheme of parameter estimation

with the estimates of state variables. This information is used to modify the control law to improve the control performance.

The procedure is shown in detail by means of two illustrative examples in the following sections, including digital simulation studies. The present scheme is of course applicable to nonlinear dynamical systems with unknown slowly time-varying parameters. However, since the determination of the control law for nonlinear systems is quite difficult, we shall treat only linear systems with unknown parameters.

## 8.5 Examples of Adaptive Control

### 8.5.1 Second Order System with an Unknown Constant Parameter

Consider a second order system described by the following equations

$$x_1(k+1) = x_1(k) + Tx_2(k), \quad (8.9)$$

$$x_2(k+1) = -Ax_2(k) + u(k) + w(k), \quad (8.10)$$

where  $x_1(k)$  and  $x_2(k)$  are state variables,  $u(k)$  is a control function to be determined, and  $w(k)$  is a Gaussian white noise with mean zero and variance  $q$ ;  $A$  is an unknown constant parameter to be estimated, and  $T$  is a known constant. It is assumed that the only one output observation is available in the form

$$z(k) = x_1(k) + v(k), \quad (8.11)$$

where  $z(k)$  is an observed signal, and  $v(k)$  is an observation noise with mean zero and variance  $r$  and is independent of the  $w(k)$ .

Now letting  $x_3(k) = A$ , we have the following estimation scheme from Eqs. (6.37) to (6.42).

(i) The estimates of state variables and unknown parameters are recursively given by

$$x_1^*(k+1) = x_1^*(k) + Tx_2^*(k) + \Delta_{11}(k+1)[z(k+1) - x_1^*(k) - Tx_2^*(k)], \quad (8.12)_1$$

$$x_2^*(k+1) = -x_2^*(k)x_3^*(k) + u(k) + \Delta_{21}(k+1)[z(k+1) - x_1^*(k) - Tx_2^*(k)], \quad (8.12)_2$$

$$x_3^*(k+1) = x_3^*(k) + \Delta_{31}(k+1)[z(k+1) - x_1^*(k) - Tx_2^*(k)], \quad (8.12)_3$$

where

$$\Delta_{i1}(k+1) = P_{i1}(k+1)/[r + P_{11}(k+1)], \quad i = 1, 2, 3. \quad (8.13)$$

(ii) The variance equations are

$$P(k+1) = F(k)M(k)F'(k) + Q(k) \quad (8.14)$$

and

$$M_{ij}(k+1) = P_{ij}(k+1) - P_{i1}(k+1)P_{j1}(k+1)/[r + P_{11}(k+1)], \quad (8.15)$$

where  $i, j = 1, 2, 3$ , and

$$Q(k) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (8.16)$$

(iii) The equivalent gain matrix becomes

$$F(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_3^*(k) & x_2^*(k) \\ 0 & 0 & 1 \end{pmatrix}. \quad (8.17)$$

(iv) The modification of the variance  $M_{33}(k)$  is

$$M_{33}(k) = \begin{cases} \text{no modification for } 0 \leq k < N_3 \\ \epsilon \text{ for } k \geq N_3 \end{cases} \quad (8.18)$$

where  $N_3$  is the first time  $k$  such that  $M_{33}(k) \leq \epsilon$ , and it should be noted that when the unknown parameter is constant, the choice of the threshold value  $\epsilon$  is naturally zero.

If the value of the parameter  $A$  is precisely known, the optimal control which minimizes the expectation of the quadratic loss function

$$E \left\{ \sum_{j=k}^{N-1} x_1^2(j+1) \right\} \quad (8.19)$$

is given by, for all  $N$ ,

$$u(k) = -\frac{1}{T} x_1(k) - (1 + A)x_2(k), \quad (8.20)$$

where  $x_1(k)$  and  $x_2(k)$  are the optimal estimates of state variables  $x_1(k)$  and  $x_2(k)$  based on  $Z_k = \{z(1), \dots, z(k)\}$ , respectively.

Even if the parameter  $A$  is unknown, the problem of optimal adaptive control can be formulated by applying the dynamic programming[5]. To the author's knowledge, however, since it is quite difficult to obtain the optimal estimates of state variables as well as those of unknown parameters, the optimal adaptive control can not be solved except for extremely simple examples[1]. The difficulty of the state and parameter estimation problems for such systems has already discussed in Chapters 4 and 6. Furthermore, the feasibility of the approximate estimation scheme has also examined by digital simulations for first and second order systems; and it is proved that sufficiently large number of

observations give plausible estimates of unknown parameters as well as those of state variables.

Accordingly, if the control interval is sufficiently large, the control law

$$u^*(k) = -\frac{1}{T} x_1^*(k) - (1 + x_3^*(k))x_2^*(k) \quad (8.21)$$

is expected to give a good performance. In this case,  $u(k)$  in Eq. (8.12)<sub>2</sub> is to be replaced by  $u^*(k)$  given by Eq. (8.21).

Digital simulation studies are carried out by using the following numerical values:

$$\begin{aligned} A &= 0.9 (\text{to be estimated}), \quad T = 0.2, \quad r = 0.005, \quad x_1(0) = 2, \\ x_2(0) &= 0, \quad A_0 = 0.5 = x_3^*(0), \quad x_1^*(0) = 0.5, \quad x_2^*(0) = 0, \\ M(0) &= I \quad (3 \times 3 \text{ unit matrix}), \end{aligned}$$

and  $q$  and  $\epsilon$  are employed as parameters of the experiment.

Fig. 8.3 displays typical tracking behaviors for the unknown parameter, where  $q = 0.09$ , and  $\epsilon = 0$  and  $0.05$ .

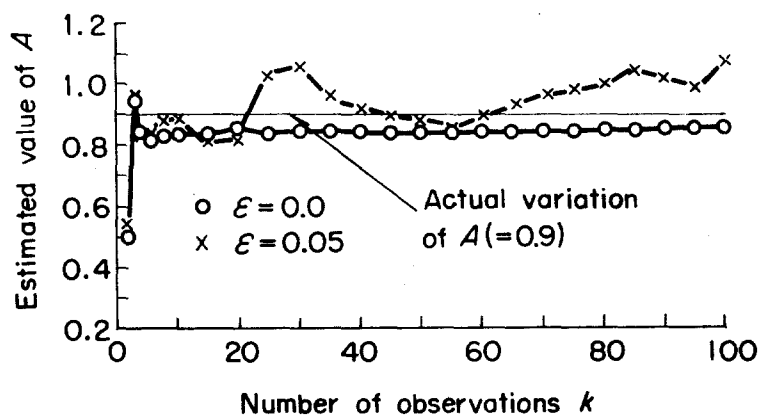


Fig. 8.3 Tracking behaviors for the unknown constant parameter  $A$ .

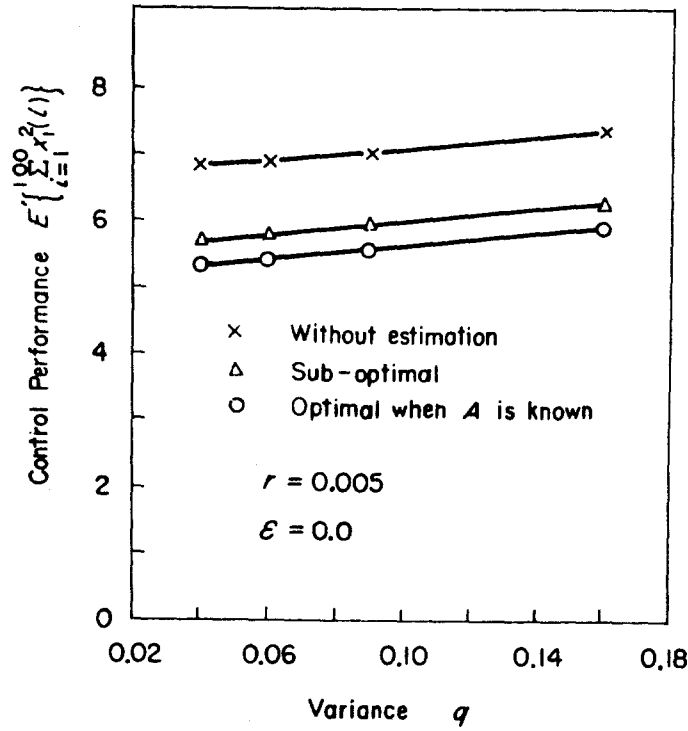


Fig. 8.4 Evaluation of control performance by three different control law, Eqs. (8.20), (8.21) and (8.23)

Fig. 8.4 displays the experimental evaluation of the control performance

$$E' \left\{ \sum_{i=1}^{100} x_1^2(i) \right\} \quad (8.22)$$

where  $E'$  denotes the average of 50 trials by use of different pseudo-random numbers. The result by use of the suboptimal control law defined by Eq. (8.21) is shown by  $\Delta$ . For the comparison of the present method the results by two different schemes are also shown; one of which is obtained by using Eq. (8.20) under the assumption that the parameter  $A$  is completely known; the other is the result obtained by the control law

$$u_a(k) = cz(k), \quad c = -3.25. \quad (8.23)$$

The feedback gain in Eq. (8.23) is determined on the basis of the a priori information of the unknown parameter A. In fact, from Eqs. (8.11) and (8.23), Eqs. (8.9) and (8.10) can be expressed as

$$x_1(k+2) + (1 + A)x_1(k+1) + (A - cT)x_1(k) = \xi(k), \quad (8.24)$$

where

$$\xi(k) = w(k) + cv(k) \quad (8.25)$$

and

$$\Phi_{\xi\xi}(z) = E\{\xi^2(k)\} = q + c^2 r. \quad (8.26)$$

Therefore, the mean square value of  $x_1(k)$  as  $k \rightarrow \infty$  can be evaluated by use of z-transform method[21] as follows.

$$E\{x_1^2\} = \frac{1}{2\pi j} \int_{\Gamma} G(z)G(z^{-1})\Phi_{\xi\xi}(z) \frac{dz}{z}, \quad (8.27)$$

where

$$G(z) = \frac{1}{z^2 + (1 + A)z + (A - cT)} \quad (8.28)$$

and  $\Gamma$  is the unit circle in the complex z-domain, and the a priori mean of the unknown parameter  $A_0 = 0.5$ . The numerical value  $c = -3.25$  is the approximate solution which minimizes  $E\{x_1^2\}$ , when  $A_0 = 0.5$ .

Fig. 8.5 displays the effect of the threshold value  $\epsilon$  on the control performance. In this case, if  $\epsilon$  is sufficiently small, it does not affect the control performance, but the parameter estimation is very much affected by the threshold value  $\epsilon$ , as shown in Fig. 8.3.

Next we shall consider a first order system with a slowly time-

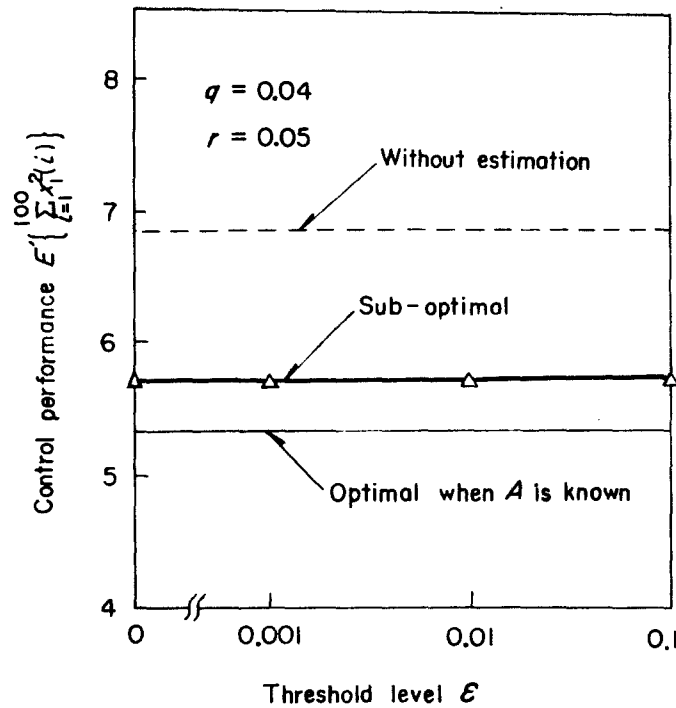


Fig. 8.5 Effect of threshold value  $\epsilon$  on the control performance, where  $A = 0.9$  (to be estimated)

varying parameter.

#### 8.5.2 First Order System with a Slowly Time-Varying Parameter

We consider a first order system described by the equation

$$x(k+1) = \alpha(k)x(k) + \beta u(k) + w(k), \quad (8.29)$$

where  $x(k)$  is a state variable,  $u(k)$  is a control function to be determined,  $w(k)$  is a Gaussian white noise with mean zero and variance  $q$ ,  $\beta$  is a known constant, and  $\alpha(k)$  is an unknown parameter in question. The output observation is a state variable corrupted by an additive noise, i.e.,



$$z(k) = x(k) + v(k), \quad (8.30)$$

where  $z(k)$  is an observed signal, and  $v(k)$  is a Gaussian white noise with mean zero and variance  $r$  and is independent of the  $w(k)$ .

Application of the present technique gives the following estimation scheme.

(i) The difference equations of the estimator are

$$\begin{aligned} x^*(k+1) = & \alpha^*(k)x^*(k) + \beta u(k) \\ & + \Delta_1(k+1)[z(k+1) - \alpha^*(k)x^*(k) - \beta u(k)] \end{aligned} \quad (8.31)$$

and

$$\alpha^*(k+1) = \alpha^*(k) + \Delta_2(k+1)[z(k+1) - \alpha^*(k)x^*(k) - \beta u(k)], \quad (8.32)$$

where

$$\Delta_1(k+1) = P_{11}(k+1)/[r + P_{11}(k+1)], \quad (8.33)_1$$

$$\Delta_2(k+1) = P_{12}(k+1)/[r + P_{11}(k+1)]. \quad (8.33)_2$$

(ii) The variance equations are given by

$$\begin{aligned} P_{11}(k+1) = & F_{11}^2(k)M_{11}(k) + 2F_{11}(k)F_{12}(k)M_{12}(k) \\ & + F_{12}^2(k)M_{22}(k) + q, \end{aligned} \quad (8.34)_1$$

$$P_{12}(k+1) = F_{11}(k)M_{12}(k) + F_{12}(k)M_{22}(k), \quad (8.34)_2$$

$$P_{22}(k+1) = M_{22}(k), \quad (8.34)_3$$

and

$$M_{11}(k+1) = rP_{11}(k+1)/[r + P_{11}(k+1)], \quad (8.35)_1$$

$$M_{12}(k+1) = rP_{12}(k+1)/[r + P_{11}(k+1)], \quad (8.35)_2$$

$$M_{22}(k+1) = P_{22}(k+1) - P_{12}^2(k+1)/[r + P_{11}(k+1)], \quad (8.35)_3$$

where  $M_{ij}$ ,  $P_{ij}$  and  $F_{ij}$  are elements of the matrices  $M$ ,  $P$  and  $F$ , respectively, and indices 1 and 2 denote the state variable and the unknown parameter, respectively.

(iii) The equivalent gain matrix is

$$F(k) = \begin{pmatrix} \alpha^*(k) & x^*(k) \\ 0 & 1 \end{pmatrix}. \quad (8.36)$$

(iv) The modification of the variance  $M_{22}(k)$  is

$$M_{22}(k) = \begin{cases} M_{22}(k) & \text{for } 0 \leq k < N_2, \\ \epsilon & \text{for } k \geq N_2, \end{cases} \quad (8.37)$$

where a random variable  $N_2$  is the first time  $k$  such that  $M_{22}(k) \leq \epsilon$ , and depends upon the sample processes  $w(k)$  and  $v(k)$ .

If the variation of the parameter  $\alpha(k)$  is completely known, the control law which minimizes the expectation of the quadratic loss function

$$E\left\{\sum_{j=k}^{N-1} x^2(j+1)\right\} \quad (8.38)$$

is given by, for all  $N$ ,

$$\hat{u}(k) = -\alpha(k)\hat{x}(k)/\beta, \quad (8.39)$$

where  $\hat{x}(k)$  is again the optimal estimate of the state variable  $x(k)$  based on  $Z_k$ .

In the present case, however, the future behavior of the parameter  $\alpha(k)$  can not be predicted in advance, so that the control law defined by

Eq. (8.39) is impossible in reality. Therefore, we again adopt the control law

$$u^*(k) = - \alpha^*(k)x^*(k)/\beta \quad (8.40)$$

which is expected to give a good performance as the previous example, if the control interval is sufficiently large.

The experimental simulations by a digital computer are carried out. The following numerical values are used in the computation.

$$\alpha(k) = 0.4 + 0.004k \text{ (to be estimated)}, \quad \beta = 1.0, \quad x(0) = 1.5,$$

$$x^*(0) = 0.5, \quad \alpha^*(0) = \alpha_0 = 0.5,$$

$$M(0) = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

and  $q$ ,  $r$  and  $\varepsilon$  are employed as the parameters of the experiment.

Fig. 8.6 displays tracking behavior for the unknown parameter  $\alpha(k)$  by use of the present scheme, where  $q = 0.04$  and  $r = 0$ .

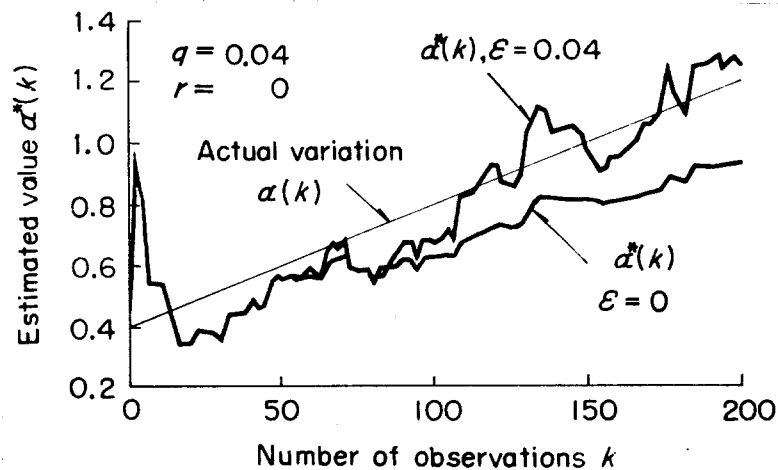


Fig. 8.6 Typical tracking behavior for a slowly time-varying parameter  $\alpha(k)$

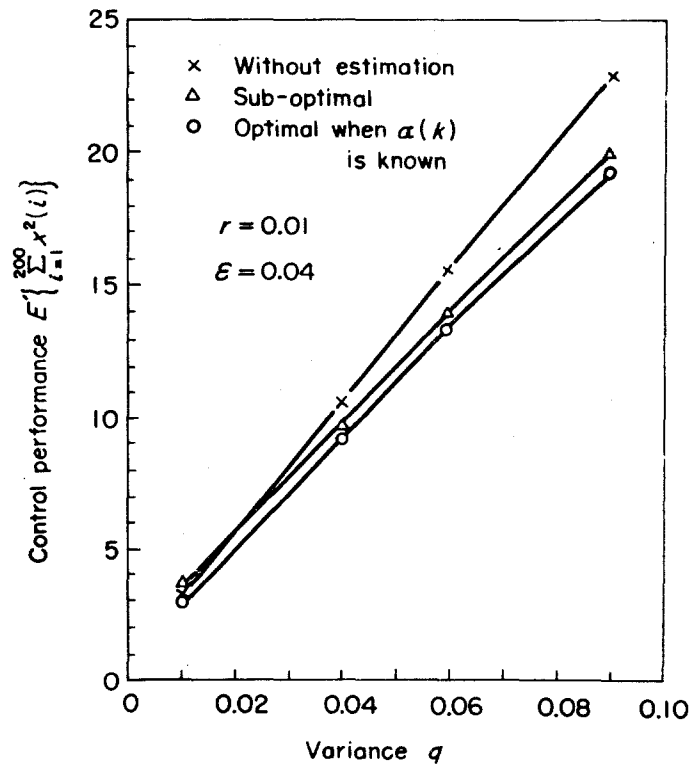


Fig. 8.7 Evaluation of the control performance by three different control laws, Eqs. (8.39), (8.40) and (8.42)

Fig. 8.7 displays the experimental evaluation of the control performance

$$E' \left\{ \sum_{i=1}^{200} x^2(i) \right\}, \quad (8.41)$$

where  $E'$  denotes the average of 50 trials for different pseudorandom numbers, and  $r = 0.01$  and  $\epsilon = 0.04$ . The result obtained by the present scheme is shown by  $\Delta$ . For the reference, the result obtained by use of Eq. (8.39) assuming that  $\alpha(k)$  is known is shown by  $o$ , and the other result by use of the control law

$$u_a(k) = -\alpha_o z(k)/\beta \quad (8.42)$$

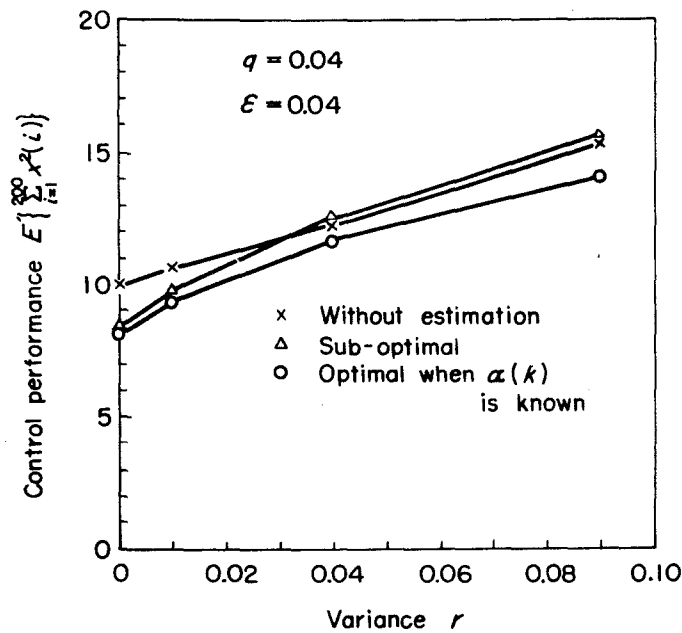


Fig. 8.8 Evaluation of the control performance by three different control laws, Eqs. (8.39), (8.40) and (8.42)

is also shown by  $\times$ . This control is formed on the basis of the a priori information under the assumption that no estimation is available.

Fig. 8.8 displays the results by three different control laws, Eqs. (8.39), (8.40) and (8.42), where  $q = 0.04$  and  $\epsilon = 0.04$ .

Fig. 8.9 displays the effect of the threshold value  $\epsilon$  on the control performance, where  $q = 0.04$  and  $r = 0$ .

The simulation studies above demonstrate that, if the observation noise variance  $r$  is small and if the disturbance level  $q$  is not very small, and if the threshold value  $\epsilon$  is properly chosen, then the control law  $u^*(k)$ , defined by Eq. (8.40), gives rather good performance. The proper choice of the threshold value  $\epsilon$  is largely dependent upon the nature of the parameter variation in question. This fact is clearly

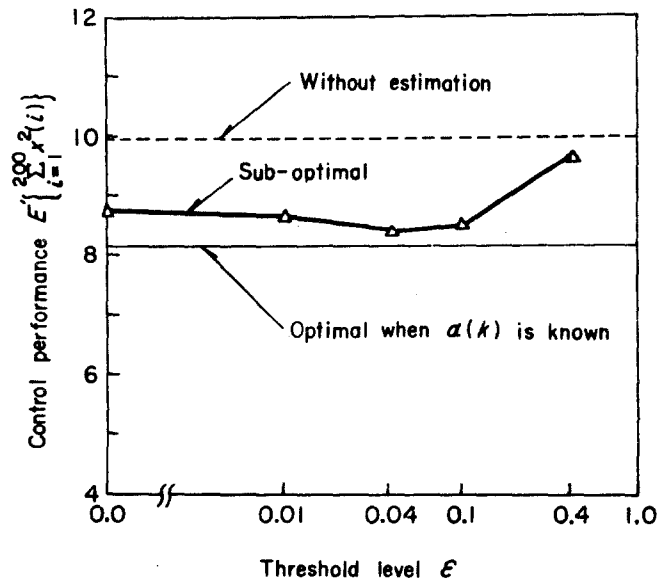


Fig. 8.9 Effect of threshold level  $\epsilon$  on the control performance

shown in Fig. 8.9.

Since the parameter variation can not be predicted in advance, it is impossible to choose a proper threshold value. However, if the parameter variation lies in some region, say  $[a, b]$ , we can choose, for example,  $\epsilon = (b - a)^2/12$ , assuming that the value of the parameter is uniformly distributed over  $[a, b]$ .

It should not be overlooked that a minimum value of the control performance evaluated by a digital computer appears around  $\epsilon = 0.04$ , as shown in Fig. 8.9. This fact shows that the threshold value has an effect on the control performance of the system with time-varying unknown parameters.

## 8.6 Concluding Remarks

In this chapter, we have considered the nonstationary parameter estimation problem for noisy discrete-time nonlinear dynamical systems. The estimation method for constant parameters is extended to include the estimation of slowly time-varying parameters by modifying the error covariance matrix. The method of the modification of the covariance matrix is presented, by which the greater weight is assigned to the latest observation. The procedure is schematically shown in Fig. 8.1.

The result is then applied to the adaptive compensation of the control performance due to the existence of unknown parameters. The procedure is explained by means of two illustrative examples. The result of digital simulation studies indicates that the proper choice of the threshold value largely depends upon the behavior of the parameter in question. It is interesting to note that the minimum value of the control performance evaluated by experiment appears around a particular value of the threshold  $\epsilon$ . This fact shows that the threshold value has an large effect on the control performance of the system with unknown parameters. At this stage, we may conclude that an appropriate value of the threshold level may be chosen through the accumulation of our practical experiment.

## REFERENCES

- [1] Aoki, M., Control of Stochastic Systems, Academic Press, New York, 1967.
- [2] Aseltine, J.A., A.R. Mancini and C.W. Sarture, "A Survey of Adaptive Control Systems," IRE Trans. on Automatic Control, PGAC-6, 102-108, Dec., 1958.
- [3] Balakrishnan, A.V., "A General Theory of Nonlinear Estimation in Control Systems," J. Math. Anal. Appl., 8, 4-30, Feb., 1964.
- [4] Bass, R.W., V.D. Norm and L. Schwartz, "Optimal Multichannel Non-linear Filtering," J. Math. Anal. Appl., 16, 152-164, Oct., 1966.
- [5] Bellman, R., Adaptive Control Processes: A Guided Tour, Princeton University Press, Princeton, N.J., 1961.
- [6] Bertram, J.E., "Control by Stochastic Adjustment," AIEE Trans. on Application and Industry, 78, 485-491, Jan., 1960.
- [7] Booton, R.C., "An Optimization Theory for Time-Varying Linear Systems, with Nonstationary Statistical Inputs," Proc. IRE, 40, 977-981, Aug., 1952.
- [8] Bucy, R.S., "Nonlinear Filtering Theory," IEEE Trans. on Automatic Control, AC-10, 198, April, 1965.
- [9] Cox, H., "On the Estimation of State Variables and Parameters for Noisy Dynamic Systems," IEEE Trans. on Automatic Control, AC-9, 5-12, Jan., 1964.



- [10] Cramer, H., Mathematical Methods of Statistics, Princeton University Press, Princeton, N.J., 1946.
- [11] Detchmندی, D.M., and R. Sridhar, "Sequential Estimation of States and Parameters in Noisy Nonlinear Dynamical Systems," Trans. ASME, Ser. D, J. Basic Eng. 86, 362-368, June, 1966.
- [12] Doob, J.L., Stochastic Processes, John Wiley, New York, 1953.
- [13] Dvoretzky, A., "On Stochastic Approximation," Proc. 3rd Berkeley Symp. on Mathematical Statistics and Probability, 1, 39-55, University of California Press, California, 1955.
- [14] Eykhoff, P., P.M.E.M. vander Grinten, H. Kwakernaak and B.P.Th. Veltman, "System Modelling and Identification," Survey paper, 3rd IFAC Congress, London, June, 1966.
- [15] Florentin, J.J., "Partial Observability and Optimal Control," J. Electronics and Control, 13, 263-279, Sept., 1962.
- [16] Fukao, T., "System Identification by Bayesian Learning Process," Proc. IFAC Tokyo Symp. on Systems Engineering for Control System Design, 1965, 137-146, March, 1966.
- [17] Gauss, K.F., Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections, Dover, New York, 1963(Reprint).
- [18] Ho, Y.C., and R.C.K. Lee, "A Bayesian Approach to Problems in Stochastic Estimation and Control," IEEE Trans. on Automatic Control, AC-9, 333-339, Oct., 1964.
- [19] Ho, Y.C., and R.C.K. Lee, "Identification of Linear Dynamic Systems," Information and Control, 8, 93-110, Feb., 1965.
- [20] Joseph, P.D., and J.T. Tou, "On Linear Control Theory," AIEE Trans. on Application and Industry, 80, 193-196, Sept., 1961.
- [21] Jury, E.I., Theory and Application of the z-Transform Method, John Wiley, New York, 1964.
- [22] Kalman, R.E., "Design of a Self-Optimizing Control Systems," Trans. ASME, Ser. D, J. Basic Eng., 80, 467-478, Feb., 1958.
- [23] Kalman, R.E., "A New Approach to Linear Filtering and Prediction Problem," Trans. ASME, Ser. D, J. Basic Eng., 82, 35-45, March, 1960.
- [24] Kalman, R.E., "Contributions to the Theory of Optimal Control," Proc. the Mexico City Conference on Ordinary Differential Equations, 1959, Bol. Soc. Mat. Mex., 102-119, 1960.

- [25] Kalman, R.E., "On the General Theory of Control Systems," Proc. of 1st IFAC Congress, Moscow, 1960, Butterworths, London, 1, 481-492, 1961.
- [26] Kalman, R.E., "New Method in Wiener Filtering Theory," Proc. of the 1st Symp. on Engineering Applications of Random Function Theory and Probability, 270-388, John Wiley, New York, 1963.
- [27] Kalman, R.E., and R.S. Bucy, "New Results in Linear Filtering and Prediction Theory," Trans. ASME, Ser. D, J. Basic Eng., 83, 95-108, March, 1961.
- [28] Kolmogorov, A.N., "Interpolation and Extrapolation of Stationary Random Sequences," Bull. Acad. Sci. URSS, Ser. Math., 5, 3-14, No. 5, 1941.
- [29] Kumar, K.S.P., "Identification of Nonlinear, Nonstationary Process," Proc. IFAC Tokyo Symp. on Systems Engineering for Control System Design, 1965, 237-243, March, 1966.
- [30] Kushner, H.J., "A Simple Iterative Procedure for the Identification of the Unknown Parameters of a Linear Time Varying Discrete Systems," Trans ASME, Ser. D, J. Basic Eng., 85, 227-235, June, 1963.
- [31] Kushner, H.J., "On the Differential Equations Satisfied by Conditional Probability Densities of Markov Processes, with Applications," J. SIAM, Ser. A, Control, 2, 106-119, No. 1, 1964.
- [32] Kushner, H.J., "Approximations to Optimal Nonlinear Filters," IEEE Trans. on Automatic Control, AC-12, 546-556, Oct., 1967.
- [33] Lee, R.C.K., Optimal Estimation, Identification, and Control, M.I.T. Press, Cambridge, Mass., 1964.
- [34] Levin, M.J., "Optimum Estimation of Impulse Response in the Presence of Noise," IRE Trans. on Circuit Theory, CT-7, 50-56, March, 1960.
- [35] Meier, L., and B.D.O. Anderson, "Performance of Sub-Optimal Linear Control Systems," Proc. IEE, 114, 124-128, Jan., 1967.
- [36] Neal, S.R., "Linear Estimation in the Presence of Errors in Assumed Plant Dynamics," IEEE Trans. on Automatic Control, AC-12, 592-594, Oct., 1967.
- [37] Nishimura, T., "On the a priori Information in Sequential Estimation Problems," IEEE Trans. on Automatic Control, AC-11, 197-204, April, 1966.

- [38] Nishimura, T., "Error Bounds of Continuous Kalman Filters and the Application to Orbit Determination Problems," IEEE Trans. on Automatic Control, AC-12, 268-274, June, 1967.
- [39] Sawaragi, Y., N. Sugai and T. Sunahara, Statistical Studies on Nonlinear Control Systems, Nippon Print. and Publishing Co., Osaka, Japan, 1962.
- [40] Sawaragi, Y., and T. Katayama, "On the Estimation of State Variables for Discrete-Time Noisy Nonlinear Dynamical Systems," Tech. Repts. Eng. Res. Inst. Kyoto Univ., No. 132, March, 1967.
- [41] Sawaragi, Y., and T. Katayama, "On the Parameter Estimation for Discrete-Time Noisy Nonlinear Dynamical Systems," Tech. Repts. Eng. Res. Inst. Kyoto Univ., No. 135, Oct., 1967.
- [42] Sawaragi, Y., and T. Katayama, "On the Estimation of Time-Varying Parameters for Noisy Nonlinear Systems and its Application to Adaptive Control," J. Japan Assoc. of Automatic Control Engineers, 12, 326-334, June, 1968.(in Japanese)
- [43] Sawaragi, Y., and T. Katayama, "On the Performance Loss in Sequential Estimation and Related Problems of Duality," Trans. of Society of Instrument and Control Engineers, 4, 248-255, Sept., 1968. (in Japanese)
- [44] Sawaragi, Y., and T. Katayama, "Performance Loss and its Improvement of Linear Filters in the Presence of Parameter Identification Errors," Trans. of Society of Instrument and Control Engineers. (to appear in Japanese)
- [45] Sherman, S., "Non-Mean-Square Error Criteria," IRE Trans. on Information Theory, IT-4, 125-126, Sept., 1958.
- [46] Soong, T.T., "On a priori Statistics in Minimum Variance Estimation Problems," Trans. ASME, Ser. D, J. Basic Eng., 87, 109-112, March, 1965.
- [47] Sorenson, H.W., "On the Error Behavior in Linear Minimum Variance Estimation Problems," IEEE Trans. on Automatic Control, AC-12, 557-562, Oct., 1967.
- [48] Sorenson, H.W., and A.R. Stubberud, "Recursive Filtering for Systems with Small but Non-negligible Non-linearities," Int. J. Control, 7, 271-280, March, 1968.
- [49] Steeg, C.W., "A Time Domain Synthesis for Optimum Extrapolators," IRE Trans. on Automatic Control, PGAC-3, 32-41, Nov., 1957.

- [50] Stratonovich, R.L., "Conditional Markov Processes," Theory of Probability and its Application, 5, 156-178, No. 2, 1960.
- [51] Stratonovich, R.L., Conditional Markov Processes and Their Applications to the Theory of Optimal Control, Elsevier Pub. Co., New York, 1968.
- [52] Tou, J.T., Modern Control Theory, McGraw-Hill, New York, 1964.
- [53] Wiener, N., Extrapolation, Interpolation and Smoothing of Stationary Time Series, John Wiley, New York, 1949.
- [54] Wiener, N., Nonlinear Problems in Random Theory, John Wiley, New York, 1958.
- [55] Wonham, W.M., "Some Applications of Stochastic Differential Equations to Optimal Nonlinear Filtering," J. SIAM, Ser. A, Control, 2, 347-369, No. 3, 1964.
- [56] Wonham, W.M., "On the Separation Theorem of Stochastic Control," J. SIAM, Ser. A, Control, 6, 312-326, No. 2, 1968.
- [57] Zadeh, L.A., "Optimum Nonlinear Filtering," J. Appl. Phys., 24, 396-404, April, 1953.
- [58] Zadeh, L.A., and J.R. Ragazzini, "An Extension of Wiener's Theory of Prediction," J. Appl. Phys., 21, 645-655, July, 1950.